Fault-Diagnosis of Grid Structures

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Abstract.

The problem of fault diagnosis in grid-connected systems is considered. A diagnosis algorithm, called DAGS and based on the PMC model, is presented. DAGS provides a diagnosis which is shown to be correct, although possibly incomplete, if the cardinality of the actual fault set is below a bound T_σ, dependent of the actual syndrome σ. A bound T independent of σ is also derived by a worst-case analysis covering the cases of triangular, square, hexagonal and octagonal grids. T is shown to be $\Theta(n^{2/3})$, where n is the size of the system, for all the grids considered.

Keywords: Fault-Tolerance, Fault Diagnosis, System-Level Diagnosis, Parallel Architectures.

1 Introduction

Fault diagnosis is of primary importance to provide high dependability in complex systems. It aims at identifying the (faulty or non-faulty) state of the units composing a system. Upon identification, faulty units may be either replaced or isolated from the rest of the system, and fault recovery or reconfiguration techniques may be used to restore a coherent state, allowing the system to resume operation, possibly with reduced performance (*graceful degradation*).

System-level diagnosis was introduced by Preparata, Metze and Chien [14] and has been deeply investigated in literature. It aims at diagnosing systems composed by units (usually processors), with the requirement that they are able to test each other by exchanging information through point-to-point bi-directional links. A system is represented by the *system graph* $G=(N,L)$, an

undirected graph where node set N represents units and edge set L represents interconnections. The cardinality[1] $n=\#N$ is called the *size* of the system. If edge (u,v) exists, units u and v are said to be *adjacent*, denoted $u \leftrightarrow v$.

In principle, the test of unit v, performed by unit u, proceeds as follows: the *testing unit u* provides a test sequence as input to the *tested unit v*, which returns an output sequence to u. Unit u compares the actual and the expected output sequences and provides a binary test outcome. The outcome is 0 (the test *passes*) if the actual and the expected results match, 1 (the test *fails*) otherwise.

Table 1: Invalidation rule in the PMC model.

The model assumes that the test has perfect fault coverage, that is, tests performed by non-faulty units are always reliable. If the testing unit is faulty, the test is not reliable and the test outcome is arbitrary. This *invalidation rule* of the PMC model, known as *symmetric invalidation*, is summarized in Table 1.

Other diagnostic models are based on comparisons rather than tests [13], or use different invalidation rules, such as the BGM model [1], which assumes test outcome 1 whenever the tested unit is faulty, regardless of the state of the testing unit.

In the following, notation $u \xrightarrow{\gamma} v$ denotes the test of unit v performed by unit u with outcome $\gamma \in \{0,1\}$. In this paper it is assumed that the tests are reciprocal and that any two units u,v with $u \leftrightarrow v$ test each other. Notation $u \xleftarrow{\delta \quad \gamma} v$ denotes both the test of unit v performed by unit u with outcome γ and the test of unit u performed by unit v with outcome δ. Given any set $N_f \subseteq N$ of faulty units (*actual fault set*), the resulting set of all test outcomes is called *syndrome*, denoted σ. As seen from Table 1, a fault set may yield different syndromes; conversely, a given syndrome may derive from different fault set.

[1] Throughout this paper $\#X$ denotes the cardinality of set X, for any X.

The syndrome is collected by an external, reliable *diagnoser* and it is decoded by a *diagnosis algorithm*. The diagnosis algorithm provides a *diagnosis* of the system by partitioning set N into set F of units declared faulty, set K of units declared non-faulty, and set $S=N-(F\cup K)$ of *suspect* units. Given any syndrome σ, the diagnosis is said to be *correct* if $F\subseteq N_f$ and $K\subseteq N-N_f$. The diagnosis is said to be *complete* if $S=\varnothing$.

A system is said *one-step t_0-diagnosable* if correct and complete diagnosis is always possible for every fault sets N_f with $\#N_f\leq t_0$, for some value of t_0. The value of t_0, called the *diagnosability* of the system, is limited above by the minimum of the node degrees in G [4]. A general one-step diagnosis algorithm is reported in [7].

The natural application of the system-level diagnosis theory is the diagnosis of massive parallel processing systems. One emerging, and perhaps more interesting, application is the "wafer-scale" VLSI testing [15]. In both cases the system is represented by a regular or quasi-regular graph. In the case of grid structures, which are the most common, the node degree is (or is limited above by) a small constant. This implies that the one-step diagnosability is also a small constant [8, 14]. Since the size of the systems tends to grow as the technology improves, the hypotheses of a number of faults limited by a small constant is unrealistic and the one-step diagnosis algorithms are inadequate. To overcome this limitation, alternative approaches have been proposed.

The *probabilistic approach* aims at providing a diagnosis, usually complete, whose correctness is evaluated by probabilistic models. Under this approach, Scheinerman [17] considered random graphs (i.e., graphs in which every test link exists with probability p) and showed that correct and complete diagnosis can be obtained with probability approaching to 1 as $n\to\infty$ if the average number of links per unit is slightly above $log(n)$. Blough et al. [2] reinforced this result by proving that $log(n)+c$ test links per unit, where c is a small constant, are necessary and sufficient to achieve asymptotically correct and complete diagnosis.

Somani and Agarwal [18] proposed a distributed approach to the diagnosis of regular structures. LaForge et. Al. [12] and Huang et. Al. [10] proposed algorithms to achieve almost correct and complete diagnoses in regular or quasi regular systems. Khanna and Fuchs [11] focused on *sequential diagnosis* and provided bounds to the *sequential diagnosability* t_S of several regular-interconnected systems. In particular, they showed that sequential diagnosis is possible in *d*-dimensional square grids if the number of faults does not exceed a bound $O\left(n^{\frac{d}{d+1}}\right)$.

The *deterministic approach* aims at providing a (possibly incomplete) diagnosis, which is proved to be correct for all syndromes deriving by fault sets of cardinality less than *T*, a *syndrome-independent* bound which is usually far above t_0. Under this approach, an algorithm for the diagnosis of grids of degree 4 (called *square grids*) has been introduced in [5, 6]. Given any syndrome σ, the algorithm also returns a *syndrome-dependent* bound T_σ with $T_\sigma \geq T$. The average of T_σ has been evaluated by means of simulation.

In this paper, the diagnosis algorithm presented in [5, 6] is generalized to grids of any degree, and the syndrome-independent bound *T* for diagnosis correctness is derived by a worst-case analysis for grids of degrees 3, 6 and 8. For the sake of brevity the analysis of octagonal grids will be reported in full detail, while the results obtained for the remaining cases will be presented synthetically, referring the reader to [3, 4, 16] for further details.

The paper is structured as follows. Section 2 gives some preliminary definitions and properties. The diagnosis algorithm is presented in Section 3, and the bound *T* for diagnosis correctness is derived in Section 4. Section 5 draws some conclusions.

2 Preliminaries

A toroidal grid of degree *d* is defined as a graph G_d^T of node set $N=\{u_{xy} \mid x,y$ integers, and $x,y=0,\ldots,L-1\}$, where *L* is a positive, even integer ($n=L^2$). Every units is connected to *d* neighbors according to the rules specified below. Depending on the number of neighbors (3, 4,

6 or 8), grids will be referred to as *triangular*, *square*, *hexagonal* or *octagonal*. The units are connected as follows:

- In toroidal, *triangular grids* unit u_{xy} is connected to units indexed by:
  - $(x,(y\pm1) \bmod L)$;
  - $((x+1) \bmod L,y)$ if both x and y are either even or odd, or to $((x-1) \bmod L,y)$ otherwise.
- In toroidal, *square grids* unit u_{xy} is connected to units indexed by $(x,(y\pm1) \bmod L)$ and $((x\pm1) \bmod L,y)$.
- In toroidal, *hexagonal grids* unit u_{xy} is connected to units indexed by:
  - $(x,(y\pm1) \bmod L)$ and $((x\pm1) \bmod L,y)$;
  - $((x\pm1) \bmod L, (y+1) \bmod L)$ if x is even, or to $((x\pm1) \bmod L, (y-1) \bmod L)$ otherwise.
- In toroidal, *octagonal grids* unit u_{xy} is connected to units indexed by $(x,(y\pm1) \bmod L)$, $((x\pm1) \bmod L,y)$ and $((x\pm1) \bmod L, (y\pm1) \bmod L)$.

Grids G_3^T, G_4^T, G_6^T and G_8^T of size 16 are shown in Figure 1, where the wrap-around connections are represented as if they were extending to the positive and negative infinity.

Simple grids G_3, G_4, G_6 and G_8 of degree 3, 4, 6 and 8 and size n, are derived from G_3^T, G_4^T, G_6^T and G_8^T of the same size by removing the wrap-around links crossing the border. This implies that the degree of units lying on the border is smaller than the degree of internal units; that is, simple grids are quasi-regular graphs.

Figure 1: Grids G_3^T (*a*), G_4^T (*b*), G_6^T (*c*), and G_8^T (*d*) of size 16.

Given grids G_d and $G_{d'}$, both of node set N, with $d'\neq d$, $G_{d'}$ is called *superimposed d'-grid* of G_d.

An *aggregate* A, of node set $\mathbf{A} \subseteq \mathbf{N}$, is defined as the connected component of the grid induced by $\mathbf{A}$. The cardinality of $\mathbf{A}$ is called the *area* of A. The set $\mathbf{B} = \{u \notin \mathbf{A} \mid \exists v \in \mathbf{A},\ u \leftrightarrow v\}$ is called the *boundary* of A, and $\#\mathbf{B}$ is its *perimeter*. Hereafter we use the word aggregate to mean both the connected subgrid A and its node set $\mathbf{A}$.

Two disjoint aggregates A_i and A_j are said to be *bridged* if $\mathbf{B}_i \cap \mathbf{B}_j \neq \emptyset$. Any unit u_{xy} belonging to $\mathbf{B}_i \cap \mathbf{B}_j$ is said to be a *bridge*.

Given any syndrome σ, aggregate A is said to be a *Z-aggregate* if $u \xleftarrow{0\ 0} v$ for every pair u,v with $u,v \in \mathbf{A}$ and $u \leftrightarrow v$.

The following definitions are borrowed from [7]. A directed path from u to v consisting of 0-labeled edges is denoted by $u \Rightarrow_0 v$. The *zero descendant sets* of u and $\mathbf{X}$ are defined as $\mathbf{D}_0(u) = \{v \in \mathbf{N}: u \Rightarrow_0 v\} \cup \{u\}$ and $\mathbf{D}_0(\mathbf{X}) = \bigcup_{u \in \mathbf{X}} \mathbf{D}_0(u)$. The *zero ancestor sets* of u and $\mathbf{X}$ are defined as $\mathbf{A}_0(u) = \{v \in \mathbf{N}: v \Rightarrow_0 u\}$ and $\mathbf{A}_0(\mathbf{X}) = \bigcup_{u \in \mathbf{X}} \mathbf{A}_0(u)$. The *disagreement sets* of u and $\mathbf{X}$ are defined as $\Delta_1(u) = \{v \in \mathbf{N}: u \xrightarrow{1} v \text{ or } v \xrightarrow{1} u\}$ and $\Delta_1(\mathbf{X}) = \bigcup_{u \in \mathbf{X}} \Delta_1(u)$.

The following properties derive immediately from the invalidation rule of the PMC model:

(a) if $u \xleftarrow{1\ 0} v$, then u is faulty;

(b) if u is known to be faulty, then any $v \in \mathbf{A}_0(u)$ must be faulty;

(c) if $u \xleftarrow{1\ 1} v$, then at least one unit between u and v is faulty;

(d) if $u \xleftarrow{0\ 0} v$, then both u and v are either faulty or non-faulty.

3 A Diagnosis Algorithm for Grid Structures

In this section we introduce the diagnosis algorithm DAGS (Diagnosis Algorithm for Grid Structures). Given any syndrome σ, DAGS is divided into three steps.

The first step (*Local Diagnosis*) performs a preliminary classification of units, partitioning N into sets F, D and Z. Set F contains faulty units, identified by exploiting properties (a) and (b). Units in set D (*dual-units*) are defined in disjoint pairs with the property that, for every pair, each unit accuses the other of being faulty. By property (c), at least one unit in every pair must be faulty. The remaining units are assigned to set Z (*zero-units*). Adjacent zero-units must test each other with outcome 0, since otherwise they would have been put into set F or D. Moreover, by property (d), they must be in the same state.

The second step (*Fault-Free Core Identification*) partitions the subgraph Z, induced by set Z, into disjoint Z-aggregates. Being adjacent zero-units, all units in a Z-aggregate must be in the same state. Letting α be the maximum of the Z-aggregate cardinalities, the *Fault-Free Core* (FFC) is defined as the union set of the Z-aggregates of cardinality α. In the following it will be proved that, under certain hypotheses, the FFC is non-empty and actually fault-free. In this step, the algorithm also asserts the syndrome-dependent bound T_σ, with the property that the diagnosis is correct if the cardinality of the actual fault set is less than T_σ.

In Step 3 (*Augmentation*) the FFC is recursively augmented with its zero-descendant set, thus identifying more non-faulty units, and set F constructed in Step 1 is augmented with units belonging to $\Delta_1(FFC)$ and to $A_0(FFC)$, which must be faulty under the rules of Table 1.

It is easily seen that the time complexity of DAGS is O(n) [16]. A formal description of the algorithm is reported in Table 2.

Given any syndrome σ, the diagnosis returned by DAGS is correct if there exists at least one Z-aggregate (that is, $\alpha>0$) and if every Z-aggregate of area α is fault-free. In fact, under these conditions the FFC defined in Step 2 is non-empty and actually fault-free, and Step 3 augments set FFC with units which are actually non-faulty and set F with units which are actually faulty.

The property of diagnosis correctness under the syndrome-dependent bound T_σ is stated by the following theorem:

Theorem 1. *Given any syndrome σ, the diagnosis returned by DAGS is correct provided $\alpha>0$ and $\#N_f<T_\sigma$*

Proof. Recalling that set D is constructed incrementally by adding disjoint pairs of units, and that at least one unit in every pair must be faulty, the number of faults in the system is at least $\#F+\#D/2$ (where F is the set of faulty units constructed in the first step). If some Z-aggregate of area α is not fault-free then the Fault-Free Core contains at least α faulty units, and the diagnosis is incorrect. However, this also implies that $\#N_f\geq\alpha+\#F+\#D/2=T_\sigma$, thus contradicting the hypothesis $\#N_f<T_\sigma$ ❑

Table 2. The DAGS algorithm.

4 Worst-case analysis of DAGS

In this section we derive a stronger bound for diagnosis correctness, named T, with the property that the diagnosis corresponding to any syndrome giving rise to at least one Z-aggregate is always correct if the originating fault set satisfies $\#N_f<T$. When referred to grid of degree d, T will be also denoted T_d. Intuitively, T should be derived as the minimum of T_σ over set Σ of all syndromes yielding at least one Z-aggregate. Under this definition, given any N_f with $\#N_f<T$ and letting σ be any admissible syndrome of N_f, the correctness of the diagnosis is ensured by Theorem 1, since $\#N_f<T\leq T_\sigma$.

Consider the partition of set N into sets F and D and Z-aggregates $Z_1,\ldots,Z_h$, defined by steps 1 and 2 of DAGS. Denoting $B_1, \ldots, B_h$ the boundaries of $Z_1, \ldots, Z_h$, and recalling that $Z_i\cap B_j=\varnothing$ for every $i,j=1,\ldots,h$, it is immediate that $F\cup D\supseteq\bigcup_{i=1,h}B_i$ and also $\#F+\#D/2\geq\frac{1}{2}\#\bigcup_{i=1,h}B_i$. The technique used to derive the syndrome-independent bound exploits the existence of boundaries of faulty and dual units separating the Z-aggregates. The number of faults implied by a faulty Z-aggregate of cardinality α is lower bounded by $\alpha+\#F+\#D/2$. Since the union set $\bigcup_{i=1,h}B_i$ of the

Z-aggregate boundaries is a subset of $F \cup D$, it follows that $\alpha + \#F + \#D/2 \geq \alpha + \frac{1}{2}\#(\bigcup_{i=1,h} B_i)$. Given α, the minimum $\#\bigcup_{i=1,h} B_i$ with the constraint that the cardinality of all the Z-aggregates is at most α is evaluated for all the possible sizes and shapes of Z-aggregates arising from syndromes which yield at least one Z-aggregate. A condition under which there exists at least one Z-aggregate is stated by the following lemma:

Lemma 1. *In a finite grid there exists at least one Z-aggregate if* $\#N_f < n/2$.

Proof. The number of zero-units in set N is given by expression $n - \#(F \cup D)$, where F and D are the set of faulty and dual units constructed in Step 1 of DAGS. This number is positive since $\#F + \#D/2 \leq \#N_f$ which implies $n > 2(\#F + \#D/2) \geq \#(F \cup D)$. This is sufficient to ensure that there exists at least one zero-unit in set N, and, consequently, at least one Z-aggregate. ❑

The approach to be used to evaluate the minimum of $\#\bigcup_{i=1,h} B_i$ for all possible sizes and shapes of Z-aggregates is similar to the one exploited in [6] to derive a syndrome-independent bound for simple, square grids. With this approach, the main difficulty stems from Z-aggregates reaching the border of the grid. Referring to octagonal grids, an example is shown in Figure 2, where Z-aggregates Z_1 and Z_2 are isomorphic but their perimeters $\#B_1$ and $\#B_2$ are different. This difficulty will be circumvented by analyzing infinite grids, to be defined in the following. The results of this analysis will be subsequently utilized to derive a bound holding for finite grids.

Figure 2: Two isomorphic Z-aggregates with different perimeters.

The infinite grid of degree *d,* where $d \in \{3,4,6,8\}$, extending grid G_d of size $n = L^2$, is defined as the graph G_d^{∞} of node set $N^{\infty} = \{\text{units } u_{xy} \mid x, y \text{ are integers}\}$. Unit u_{xy} is placed at coordinates (x, y) in the plane. The units are connected as follows:

- In G_3^{∞}, unit u_{xy} is connected to units indexed by:

- ✓ $(x, y\pm1)$;
- ✓ $(x+1, y)$ if both x and y are either even or odd, or to $(x–1, y)$ otherwise.

- In G_4^∞, unit u_{xy} is connected to units indexed by $(x, y\pm1)$ and $(x\pm1, y)$.
- In G_6^∞, unit u_{xy} is connected to units indexed by:
  - ✓ $(x, y\pm1)$ and $(x\pm1, y)$;
  - ✓ $(x\pm1, y+1)$ if x is even, or to $(x\pm1, y–1)$ otherwise.
- In G_8^∞, unit u_{xy} is connected to units indexed by $(x, y\pm1)$, $(x\pm1, y)$ and $(x\pm1, y\pm1)$.

Set $N^\infty–N$, which is composed by *dummy units*, is partitioned into the finite set B_0 (the boundary of N) and the infinite set $N^+=N^\infty–(N\cup B_0)$ (Figure 3). The outcomes of tests involving dummy units (*dummy tests*) are defined as follows: units in N^+ test each other with outcome 0 and test units in B_0 with outcome 1; units in B_0 test each other with arbitrary outcome, and test units in $N^\infty–B_0$ with outcome 0; units in N test units in B_0 with outcome 1.

Figure 3: The infinite grid G_8^∞.

Given the outcomes of dummy tests as defined above, the first step of DAGS would classify every unit in B_0 as faulty, while units in N^+ would be classified as zero-units and subsequently assigned to a unique Z-aggregate, henceforth denoted Z_0. The actual implementation of DAGS may limit consideration to units in N, since units in $N^\infty–N$ may be classified by "brute force". Given any syndrome, the combined result of steps 1 and 2 of the diagnosis algorithm and of the "brute force" classification of units in sets B_0 and N^+ is a partition of N^∞ into set F of units declared faulty ($F\supseteq B_0$), set D of dual-units ($D\subseteq N$), and Z-aggregates $Z_0, Z_1, \ldots, Z_h$ of node sets $Z_0, Z_1, \ldots, Z_h$, where $Z_0=N^+$ is infinite and every Z_i ($1\le i\le h$, $Z_i \subseteq N$) is finite. The existence of at least one finite Z-aggregate is guaranteed under the condition of Lemma 1.

Let Z_i ($i \geq 1$) be a finite Z-aggregate, and B_i its boundary. Consider the subgrid of node set $N^\infty - (Z_i \cup B_i)$, and the connected components of this subgrid. Since $Z_i \cup B_i \subseteq (N \cup B_0)$ and $N \cup B_0$ is finite, a unique connected component of this subgrid, denoted E_i, is infinite, while the remaining connected components, denoted $H_{i1},\ldots,H_{iq}$ ($q \geq 0$), are finite.

Let C_i be the boundary of E_i; it is immediate that $C_i \subseteq B_i$ and every $u_{vw} \in C_i$ is adjacent to both E_i and Z_i. The infinite aggregate E_i is called the *exterior* of C_i, and the subgrid I_i of node set $I_i = N^\infty - (E_i \cup C_i) = Z_i \cup (\bigcup_{k=1,q} H_{ik}) \cup (B_i - C_i)$ is called the *interior* of C_i. Observe that C_i separates I_i, and hence Z_i, from E_i, since every path going from any $u_{xy} \in I_i$ to any $u_{vw} \in E_i$ must traverse C_i. Set C_i is called the *circumscribing set* of Z_i. Subgrids $H_{i1},\ldots,H_{iq}$ are called *holes* in I_i. Every hole H_{ip} ($1 \leq p \leq q$) is an aggregate, whose boundary is denoted C_{ip}: it is immediate that $C_{ip} \subseteq B_i$. The subgrid E_{ip} of node set $N^\infty - (H_{ip} \cup C_{ip})$ is the exterior of C_{ip}, and H_{ip} is the interior. The boundary of the infinite Z-aggregate Z_0 is B_0 and the subgrid of node set $N^\infty - (Z_0 \cup B_0) = N$ is a unique, finite aggregate denoted H_{01}, which is the unique hole in N^∞.

To clarify the preceding definitions, consider the octagonal grid of Figure 4 and assume a syndrome yielding Z-aggregate Z_i: C_i is the circumscribing set, E_i is the exterior, and I_i of node set $Z_i \cup H_{i1} \cup (B_i - C_i)$ is the interior. Subgrid H_{i1} is a hole in I_i and C_{i1} is the circumscribing set of H_{i1}.

Figure 4: Z-Aggregate Z_i ($i \geq 1$).

Let aggregate A of area α be either a Z-aggregate Z_i or a hole H_{ip}, and let C be the circumscribing set of A, as defined above. The cardinality of C is related to the area α by the following lemma.

Lemma 2. *For any finite aggregate A of area α in G_d^∞, the cardinality $\#C$ of the*

circumscribing set C of A is given by:

- $\#C \geq \sqrt{6\alpha}$ *if d=3;*
- $\#C \geq 2 + 2\sqrt{2\alpha - 1}$ *if d=4;*
- $\#C \geq \sqrt{12\alpha - 3} + 3$ *if d=6;*
- $\#C \geq 4\sqrt{\alpha} + 4$ *if d=8.*

Proof. The proof of this lemma is based upon different geometric constructions for different degrees of the grid. The proof for d=8 is reported in the Appendix. For the cases of d=3, d=4 and d=6, the reader is referred to [3, 6, 16]. ❑

Consider an infinite grid of any degree and assume $\#N_f \leq n/2$; under this hypothesis there exists at least one Z-aggregate other than Z_0; that is, $h \geq 1$. For every Z_i, $1 \leq i \leq h$, consider sets B_i, C_i, I_i, E_i as defined above, and holes H_{i1}, …, H_{iq} ($q \geq 0$) with the respective circumscribing sets C_{i1}, …, C_{iq}. Observe that for every $j \neq i$, $1 \leq j \leq h$, must be either $Z_j \subseteq E_i$ or $Z_j \subseteq H_{ip}$ for some p, $1 \leq p \leq q$. Recalling that $B_i \supseteq C_i \cup \bigcup_{p=1,q} C_{ip}$ for every i, $0 \leq i \leq h$, the cardinality of $\bigcup_{i=0,h} B_i$ is evaluated as follows:

1. For every Z_i $1 \leq i \leq h$,
   1.1. the contribution of B_i to $\bigcup_{i=0,h} B_i$ is approximated by the cardinality of $C_i \cup \bigcup_{p=1,q} C_{ip} \subseteq B_i$;
   1.2. the cardinality of C_i and of every C_{ip} is approximated by the lower bound stated by Lemma 2.
   1.3. observing that any $u_{xy} \in C_i$ may also be in the circumscribing set C_j ($j \neq i$) of some $Z_j \subseteq E_i$ (Figure 5), thus contributing to $\bigcup_{i=0,h} B_i$ as a member of multiple circumscribing sets,

every $u_{xy} \in C_i$ is assigned a weight w_{xy}, to be defined in the following, with the property that the sum of the contributions of u_{xy} to $\bigcup_{i=0,h} B_i$ as a member of different circumscribing sets does not exceed 1.

1.4. similarly, observing that any $u_{xy} \in C_{ip}$ may also be in the circumscribing set C_j ($j \neq i$) of some $Z_j \subseteq H_{ip}$ $1 \leq p \leq q$ (Figure 5), thus contributing to $\bigcup_{i=0,h} B_i$ as a member of multiple circumscribing sets, the contribution of u_{xy} as a member of C_{ip} is set to a weight $\overline{w}_{xy}$, with the property that the sum of all contributions does not exceed 1.

1.5. observing that aggregates in the family E_i, H_{i1}, …, H_{iq} may be bridged, and that every u_{xy} which is a bridge would be considered multiple times as a member of $\bigcup_{i=0,h} B_i$, multiple contributions are removed using a technique to be defined in the following.

2. The boundary B_0 of the infinite Z-aggregate Z_0 is easily evaluated by inspection. The finite grid of node set N is a hole in N^{∞}, denoted H_{01}, circumscribed by $C_{01} = B_0$, and, for every $1 \leq j \leq h$, $Z_j \subseteq H_{01}$. To account for the circumstance that every $u_{xy} \in C_{01}$ may also be in the boundary of some $Z_j \subseteq H_{01}$, u_{xy} is assigned a weight $\overline{w}_{xy}$, to be defined in the following.

Figure 5: Z-aggregates, holes and bridges.

The implementation of this technique to evaluate $\bigcup_{i=0,h} B_i$ is dependent, to some extent, on the grid degree. An evaluation of $\bigcup_{i=0,h} B_i$ is provided by Lemma 3 for the case of octagonal grids and by Lemmas 4,5 and 6 for the cases of d=6, d=4 and d=3, respectively. Preliminary results needed to prove Lemma 3 are given by lemmas 3.1, 3.2, 3.3 and 3.4.

Lemma 3.1. *Given any three aggregates A_1, A_2, A_3 in G_8^{∞} and their boundaries B_1, B_2, B_3, there exist at most two distinct units belonging to $B = B_1 \cap B_2 \cap B_3$.*

Proof. Consider the subgrid X of G_8^{∞} induced by node set $B \cup A_1 \cup A_2 \cup A_3$, and assume that, contrary to the thesis, there exist distinct units u_1, u_2, u_3 belonging to $B = B_1 \cap B_2 \cap B_3$.

Preliminarily we construct a plane subgrid X^p of X by removing edges which intersect in the "natural" embedding of X (Figure 1d), while preserving the connections between units in the aggregates and between units in the aggregates and units in B.

Let (u,v) and (u',v') be intersecting edges in the "natural" embedding of X. It is immediate that such edges must be diagonal edges connecting units u,v,u',v' arranged in a square. Considering that units in $\{u,v,u',v'\}$ are pairwise adjacent and aggregates A_1, A_2, A_3 are separated by their boundaries, there exists at most one A_i (i=1..3) such that $\{u,v,u',v'\} \subseteq B \cup A_i$. Therefore all the possible cases, up to symmetries, are shown in Figure 6.

Figure 6: Pruning of X: dotted lines denote the edges which are removed.

The *pruning* of X is as follows:

Cases a), b), d) :

Remove the diagonal edge(s) connecting units in B. This way, intersecting edges are removed while preserving connections between units in A_i and between A_i and B.

c) If $v,v' \in A_i$ and $u,u' \in B$ then remove both diagonal edges. Units u and u' remain connected to A_i along vertical edges, and A_i remains connected, since the edges connecting units in A_i are not affected by the construction.

e) If $v,v',u' \in A_i$ and $u \in B$, then remove edge (u',v'). A_i remains connected, since units u',v' are connected to u along horizontal and vertical edges; moreover, u remains connected to A_i along edge (u,v).

f) If $u,v,u',v' \in A_i$ then remove both diagonal edges. The pruning only affects A_i which remains connected, since horizontal and vertical edges are retained.

The subgrid X^p resulting from the preceding construction has the following properties:

- X^p is planar, since all intersecting edges in the "natural" embedding of X are removed;
- subgrids of X^p of node set A_i (i=1..3) are aggregates;
- in X^p, units in B are adjacent to every A_i (i=1..3), since the construction does not remove edges connecting to A_i units in B.

Figure 7: Graph deriving from contracting A_1, A_2 and A_3 in graph X^p.

Consider the graph derived from X^p by contracting aggregates A_1, A_2, A_3 into units a_1, a_2, a_3 (Figure 7); this graph must be connected and planar, since X^p is connected and planar [9]. However, the graph of Figure 7 is the Kuratowski's graph $K_{3,3}$ [9]. This implies that X^p could not be planar, resulting in a contradiction. ❑

Let A (either a Z-aggregate Z_i or an hole H_{ip}) be a finite aggregate in G_8^∞ and consider the circumscribing set C of A. The following lemma exploits Lemma 3.1 to derive a characterization of set C.

For arbitrary $u_{xy} \in C$, consider set Q^8_{xy} of units which are adjacent to u_{xy} in G_8^∞. Set $Q^4_{xy} \subset Q^8_{xy}$ is defined as the set of units which are adjacent to u_{xy} in the superimposed 4-grid of G_8^∞, i.e. in G_4^∞.

Lemma 3.2. *The units in the circumscribing set C of any finite aggregate A in G_8^∞ are traversed by a simple circuit in the superimposed 4-grid G_4^∞.*

Proof. Preliminarily, it will be proved that, for any unit $u_{xy} \in C$, there exist at least two distinct units $u_{x'y'}$, $u_{x''y''} \in C \cap Q^4_{xy}$. In fact, assume by contradiction that $k = \#(C \cap Q^4_{xy}) \leq 1$. All the units in $P = Q^4_{xy} - C$ are connected by a path in Q^8_{xy}, which, up to symmetries and rotations, is shown

in Figure 8 (*a*) for k=0 and (*b*) for k=1. As this path does not cross C, all units in P must belong to either E or I. Without loss of generality, assume $P \subseteq I$. Since every unit in $Q^8_{xy} - P$ is adjacent to at least one unit in P, it follows that no unit in Q^8_{xy} belongs to E since otherwise this unit would be adjacent to I. This is a contradiction, because u_{xy}, being in C, is adjacent to both I and E.

Figure 8: Path in Q^8_{xy} for k=0 (*a*)) and k=1 (*b*)).

On the other hand, the number of units in C which are adjacent to u_{xy} in Q^4_{xy} is at most two. In fact, consider the trivial aggregate A_1 of node set $\{u_{xy}\}$: every unit in C, which is adjacent to A_1, is also adjacent to both I and E. Letting A_2=I and A_3=E, from Lemma 3.1 the number of such units is at most 2.

From the preceding results, for every unit $u_{xy} \in C$ there exist exactly two distinct units in C which are adjacent to u_{xy} in G_4^{∞}. This means that every $u_{xy} \in C$ is traversed by a simple path in G_4^{∞}, which must be a circuit, since set C is finite. Suppose that set C is split into multiple simple circuits. Given that every unit in C must be adjacent to at least one unit in A, every such circuit must circumscribe some units in A. If true, this would imply that subgrid A is composed by multiple separated components, thus contradicting the fact that aggregate A is connected. This proves that the simple circuit circumscribing A must be unique. ❑

The length of the circuit in G_4^{∞} traversing C will be denoted λ. From the preceding lemma, $\lambda = \#C$.

Consider set Q^8_{xy} of units which are adjacent to an arbitrary unit $u_{xy} \in C$. By Lemma 3.2, C is a simple circuit in G_4^{∞}. Consider the unique ordering of units in C, defined traversing C clockwise. Given $u_{xy} \in C$, denote $c^-(u_{xy})$ and $c^+(u_{xy})$ the units in C preceding and following u_{xy} in this ordering. This partitions set Q^8_{xy} into subsets (N_{xy}, M_{xy}, $\{c^+(u_{xy})\}$,$\{c^-(u_{xy})\}$), where N_{xy}

and M_{xy} are defined as follows:

N_{xy}={$u \in Q^8_{xy}$: u belongs to the path in Q^8_{xy} going counterclockwise from $c^-(u_{xy})$ to $c^+(u_{xy})$ };

M_{xy}={$u \in Q^8_{xy}$: u belongs to the path in Q^8_{xy} going clockwise from $c^-(u_{xy})$ to $c^+(u_{xy})$ }.

This partition of Q^8_{xy} is called the *local pattern* of u_{xy} with respect to C. Observe that $N_{xy} \cap I \neq \emptyset$ and $M_{xy} \cap E \neq \emptyset$. Moreover, every $u_{xy} \in C$ has a unique local pattern, since units $c^+(u_{xy})$ and $c^-(u_{xy})$ are unique. Up to rotation, there exist three classes of local patterns, shown in Figure 9.

Figure 9: Local patterns.

Consider an arbitrary unit $u_{xy} \in C$ and its unique local pattern. In general, this unit might also belong to the circumscribing set of one or more aggregates in the exterior of the circumscribing set of A. As it was anticipated in 1.3, u_{xy} is assigned a weight w_{xy} in such a way that the sum of the weights assigned to u_{xy} as a member of all circumscribing sets to which it belongs is at most one. The weight w_{xy} depends on the maximum number of distinct aggregates which have non-empty intersections with M_{xy}. It is immediate that this number is $\lceil \#M_{xy}/2 \rceil$. Weight w_{xy} of unit $u_{xy} \in C$ is defined as $w_{xy}=n_{xy}/(n_{xy}+m_{xy})$, where $n_{xy}=\lceil \#N_{xy}/2 \rceil$ and $m_{xy}=\lceil \#M_{xy}/2 \rceil$. Similarly, to the purpose anticipated in 1.4, the *complementary weight* $\overline{w}_{xy}$ of u_{xy} is defined as $\overline{w}_{xy}=1-w_{xy}=m_{xy}/(n_{xy}+m_{xy})$.

Consider the line l^-_{xy} going from the point at the coordinates of $c^-(u_{xy})$ to the point (x,y), and line l^+_{xy} going from the latter point to the point at the coordinates of $c^+(u_{xy})$ (Figure 9): these define an angle β_{xy}, positive clockwise. The segment of l^+_{xy} beginning at the coordinates (x,y) and ending at the coordinates of $c^+(u_{xy})$ is a side of a polygon whose vertices are units of C, and

angle β_{xy} is an external angle of the polygon. Segments defined by lines which intersect at u_{xy} with angle 0 are considered distinct sides: this means that the polygon has $\lambda=\#C$ sides and the same number of external angles.

Table 3. Classification of local patterns for octagonal grids.

The weights and the angles for the three classes of local patterns are displayed in Table 3. The number of local patterns of class j ($1\le j\le 3$) in the circumscribing set C is denoted ν_j. Of course, $\nu_1+\nu_2+\nu_3=\lambda$.

Given the circumscribing set C of any aggregate A, the weights w and $\overline{w}$ of C are defined as $w=\sum_{u_{xy}\in C} w_{xy}$ and $\overline{w}=\sum_{u_{xy}\in C}\overline{w}_{xy}$, respectively. The following lemma provides lower bounds to weights w and $\overline{w}$:

Lemma 3.3. *In G_8^∞, given any finite aggregate A of area α and its circumscribing set C, the weights w and $\overline{w}$ of C are given by $w\ge 2\sqrt{\alpha}+1$ and $\overline{w}\ge 2\sqrt{\alpha}+3$.*

Proof. The circuit C defines a polygon with λ vertices and λ exterior angles, whose sum is 2π. From Table 3 we have $(\nu_1-\nu_2)\cdot\pi/2=2\pi$, that is: $\nu_1-\nu_2=4$. By adding weights w_{xy} and $\overline{w}_{xy}$ of every unit in C we obtain:

$$w=\tfrac{1}{4}\nu_1+\tfrac{3}{4}\nu_2+\tfrac{1}{2}\nu_3 \text{ and } \overline{w}=\tfrac{3}{4}\nu_1+\tfrac{1}{4}\nu_2+\tfrac{1}{2}\nu_3. \qquad (1)$$

From $\lambda=\nu_1+\nu_2+\nu_3$, we derive $\nu_3=\lambda-\nu_1-\nu_2$, which, along with $\nu_1=\nu_2+4$, can be replaced in (1) yielding:

$$w=\tfrac{1}{2}\lambda-1 \text{ and } \overline{w}=\tfrac{1}{2}\lambda+1. \qquad (2)$$

From Lemma 2, $\lambda\ge 4\sqrt{\alpha}+4$, which, replaced in (2), yields: $w\ge 2\sqrt{\alpha}+1$ and $\overline{w}\ge 2\sqrt{\alpha}+3$. ❑

Given Z-aggregate Z_i ($i\ge 1$), consider aggregates E_i, $H_{i1},\ldots, H_{iq}$ as defined above, which may be

bridged. For the ease of notation define $H_{i0}=E_i$, and let $C_{i0}=C_i$.. The number of bridges among aggregates in the family $H_{i0},\ldots, H_{iq}$ is bounded as follows.

Lemma 3.4. *In G_8^∞, the number γ of bridges among aggregates in the family $H_{i0},\ldots, H_{iq}$ satisfies $\gamma\leq 2q$.*

Proof. Preliminarily, it will be shown that the number of aggregates in the family $H_{i0},\ldots, H_{iq}$ sharing a bridge is at most 2. In fact, assume without loss of generality that u_{xy} is a bridge between H_{ip} and H_{ik} ($0\leq p,k\leq q$). Unit u_{xy} must be adjacent to Z_i, because it belongs to C_{ip}. Furthermore, aggregates Z_i, H_{ip} and H_{ik} must be separated.

Figure 10: Z-aggregate Z_i and holes H_{ip}, H_{ik} and H_{ij}.

Assume by contradiction that there exist another aggregate H_{ij} ($j\neq p$, $j\neq k$, $0\leq j\leq q$) sharing the bridge. Up to symmetries, the only admissible position of units in Z_i, H_{ip}, H_{ik} and H_{ij} in Q^8_{xy} is shown in Figure 10. Unit u_{vw} must be in the boundary of H_{ij} and H_{ik}, and, by hypothesis, must be in the boundary of Z_i. Therefore one of the units in position 1, 2 or 3 must belong to Z_i. However this is not the case since all these units are adjacent to holes H_{ik}, H_{ij}, thus contradicting the fact that Z_i and the holes are separated.

Given this preliminary result, consider Z_i and define Y_k ($0\leq k\leq q$) as the set of all bridges connecting aggregates $H_{i0}, \ldots, H_{ik}$ ($\gamma=\#Y_q$), and X_k as the subgrid of G_8^∞ of node set $H_{i0}\cup\ldots\cup H_{ik}\cup Y_k$. Without any loss of generality, assume that H_{ik} is bridged with X_{k-1}, for $k=1,..,q$. By finite induction, it will be proved that $\gamma\leq 2q$.

The thesis is true if $q=0$, since $X_q=H_{i0}$ and $\gamma=0$. Assuming that the thesis holds for $q=k$, it also holds for $q=k+1$. In fact, consider aggregate H_{ik+1} which is bridged to X_k, and assume by contradiction that u_{xy}, $u_{x'y'}$, $u_{x''y''}$ are bridges in Y_{k+1}. Every such bridge is adjacent to X_k and to

H_{ik+1}. Being included in the boundary of X_q, it is also adjacent to Z_i. Renaming $X_k=A_1$, $H_{ik+1}=A_2$ and $Z_i=A_3$, from Lemma 3.1 units $u_{xy},u_{x'y'},u_{x''y''}$ cannot be in the intersection of the respective boundaries, thus yielding a contradiction. It is concluded that must be $\gamma\leq 2q$. ❑

Observe that, taking q=1 in the preceding inequality, yields $\gamma\leq 2$: this means that the number of bridges between any two aggregates is at most two.

Using the preceding results, Lemma 3 provides a lower bound to $\#\bigcup_{i=0,h}B_i$ for the case of octagonal grids.

Lemma 3. *In the infinite grid G_8^∞, let Z_0, Z_1, …, Z_h be the Z-aggregates defined by DAGS, B_i be the boundary of Z_i and α_i be the area of Z_i for i=1,…,h. If $\#N_f<n/2$, then:*

$$\#\bigcup_{i=0}^{h}B_i \geq \sum_{i=1,h}\left(2\sqrt{\alpha_i}+1\right)+2L+3.$$

Proof. Under the hypothesis $\#N_f<n/2$, there exists at least one Z-aggregate other than Z_0; that is, $h\geq 1$. For every Z_i, $1\leq i\leq h$, consider B_i, C_i, I_i, E_i as defined above, and holes H_{i1}, …, H_{iq} ($q\geq 0$) with the respective circumscribing sets C_{i1}, …, C_{iq}. Recalling that $\bigcup_{i=0,h}B_i\supseteq\bigcup_{i=0,h}(C_i\cup\bigcup_{p=1,q}C_{ip})$, the cardinality of $\bigcup_{i=0,h}B_i$ will be lower bounded by the cardinality of $\bigcup_{i=0,h}(C_i\cup\bigcup_{p=1,q}C_{ip})$. As it was anticipated earlier in this section, the contribution of every C_i, i=1…h, is determined as follows:

a) Every $u_{xy}\in C_i$ ($1\leq i\leq h$) has a unique local pattern of weight w_{xy}. For the ease of notation rename Z_i as Z_{i1} and C_i as C_{i1}. Let p_{xy}^{i1} be the unique local pattern of u_{xy} in C_{i1} and $w_{xy}^{i1}=\frac{n_{xy}^{i1}}{n_{xy}^{i1}+m_{xy}^{i1}}$ be the weight of p_{xy}^{i1}. Beside Z_{i1}, unit $u_{xy}\in C_{i1}$ may be adjacent to Z-aggregates $Z_{i2}\subseteq E_i$,…, $Z_{ik}\subseteq E_i$, where $k\leq m_{xy}^{i1}+1$, because among units in E_i being adjacent to u_{xy}, at most m_{xy}^{i1} might belong to distinct aggregates Z_{ij}, $2\leq j\leq k$.

Since unit u_{xy} will contribute to $\bigcup_{i=0,h} B_i$ as a member of C_{i1}, C_{i2}, …, C_{ik}, and such contributions will be evaluated independently using the same technique, we need to ensure that the sum of the contributions of u_{xy} does not exceed 1. To this purpose, consider Z_{i1}, Z_{i2},…, Z_{ik} and the respective C_{i1},…, C_{ik}. The contribution of u_{xy} to C_{ij} (j=1,…,k), is set to the weight w_{xy}^{ij} of u_{xy} in C_{ij}, where $w_{xy}^{ij} = \frac{n_{xy}^{ij}}{n_{xy}^{ij} + m_{xy}^{ij}}$. We claim that $\sum_{j=1,k} w_{xy}^{ij} \leq 1$. In fact, for any j,l=1, …, k, $j \neq l$, must be $N_{xy}^{il} \subseteq M_{xy}^{ij}$, and thus $\sum_{j=1,k} n_{xy}^{ij} \leq n_{xy}^{il} + m_{xy}^{il}$, for any l=1,…,k. Among the local patterns $p_{xy}^{i1},...,p_{xy}^{ik}$ of u_{xy} in C_{i1},…, C_{ik}, consider pattern p_{xy}^{it} ($1 \leq t \leq k$) for which $n_{xy}^{it} + m_{xy}^{it}$ is minimum. We thus have $\sum_{j=1,k} \frac{n_{xy}^{ij}}{n_{xy}^{ij} + m_{xy}^{ij}} \leq \frac{1}{n_{xy}^{it} + m_{xy}^{it}} \sum_{j=1,k} n_{xy}^{ij} \leq 1$.

b) Consider any hole in the interior of C_i, denoted H_{ip} ($1 \leq p \leq q$), the respective circumscribing set C_{ip}, and the exterior E_{ip} of C_{ip}. It is immediate that $Z_i \subseteq E_{ip}$ and that there may exist some Z-aggregate Z_j ($j \neq i$) with $Z_j \subseteq H_{ip}$. Assume initially that H_{ip} is not bridged to any H_{ir} ($1 \leq r \leq q$, $r \neq p$), nor to E_i. This limitation will be removed in c). Define, for the ease of notation, $C_{ip}^1 = C_{ip}$, and assume that $u_{xy} \in C_{ip}^1$ is shared by the circumscribing sets $C_{ip}^2,...,C_{ip}^k$ of Z-aggregates $Z_{ip}^2,...,Z_{ip}^k$, with $Z_{ip}^2 \subseteq H_{ip},...,Z_{ip}^k \subseteq H_{ip}$. Consider the unique local pattern p_{xy}^{i1} of u_{xy} in C_{ip}^1 and weight $\bar{w}_{xy}^{i1} = \frac{m_{xy}^{i1}}{n_{xy}^{i1} + m_{xy}^{i1}}$ of p_{xy}^{i1} and observe that among units in N_{xy}^{i1} being adjacent to u_{xy}, at most n_{xy}^{i1} might belong to distinct aggregates Z_{ij}, $2 \leq j \leq k$.

The contribution of u_{xy} to the weight w_{ip}^{j} of C_{ip}^{j} is $w_{xy}^{ij} = \frac{n_{xy}^{ij}}{n_{xy}^{ij} + m_{xy}^{ij}}$, where w_{xy}^{ij} is the weight of local pattern p_{xy}^{ij} of u_{xy} in C_{ip}^{j}, for j=2,..,k. This considered, the contribution of u_{xy} as a member of C_{ip}^1 is defined as $\bar{w}_{xy}^{i1} = \frac{m_{xy}^{i1}}{n_{xy}^{i1} + m_{xy}^{i1}}$. We claim that $\bar{w}_{xy}^{i1} + \sum_{j=2,k} w_{xy}^{ij} \leq 1$. In fact, for

every j,l with $2\le j,l\le k$, $j\ne l$, from $Z_{ip}^{j}\subseteq H_{ip}$ and $Z_{ip}^{j}\cap Z_{ip}^{l}=\varnothing$ it follows

$m_{xy}^{i1}+\sum_{j=2,k} n_{xy}^{ij}\le m_{xy}^{i1}+n_{xy}^{i1}$ and $m_{xy}^{i1}+\sum_{j=2,k} n_{xy}^{ij}\le m_{xy}^{il}+n_{xy}^{il}$, for any l=2,...,k. Letting p_{xy}^{it} ($1\le t\le k$) be the local pattern for which $n_{xy}^{it}+m_{xy}^{it}$ is minimum, we have

$$\frac{m_{xy}^{i1}}{n_{xy}^{i1}+m_{xy}^{i1}}+\sum_{j=2,k}\frac{n_{xy}^{ij}}{n_{xy}^{ij}+m_{xy}^{ij}}\le\frac{m_{xy}^{i1}+\sum_{j=2,k} n_{xy}^{ij}}{n_{xy}^{it}+m_{xy}^{it}}\le 1\,.$$

c) Consider now the possible existence of bridges among aggregates E_i, H_{i1}, …, H_{iq}. For the ease of notation define $H_{i0}=E_i$, and let $C_i=C_{i0}$. The contribution of C_{i0}, C_{i1},…, C_{iq} to $\bigcup_{i=0,h} B_i$ is the sum of weights w_{i0}, w_{i1},…, w_{iq}, less the multiple contribution due to bridges between aggregates in the family E_i, H_{i1}, …, H_{iq}. From Lemma 3.4, the number of bridges in this family is $\gamma\le 2q$. Let u_{xy} be a bridge shared by C_{ip} and C_{ir} ($0\le p, r\le q$, $p\ne r$), and consider weights assigned to u_{xy} in C_{ip} and C_{ir}. Both weights are less than 1, but their sum may exceed 1. In order to ensure that u_{xy} contributes at most 1 to $\bigcup_{i=0,h} B_i$, it is sufficient to subtract 1 from $w_{ip}+w_{ir}$. This subtraction is needed for all bridges, which total up to γ.

From a), b) and c), and considering Lemma 3.3, we obtain the following inequality for $w_i=\#(C_i\cup\bigcup_{p=1,q} C_{ip})$, with i=1..h:

$$w_i\ge 2\sqrt{\alpha_i}+1+\sum_{p=1}^{q}\left(2\sqrt{\#H_{ip}}+3\right)-\gamma\,. \qquad (3)$$

By replacing $\gamma\le 2q$ into (3) we obtain:

$$w_i\ge 2\sqrt{\alpha_i}+1+\sum_{p=1}^{q}\left(2\sqrt{\#H_{ip}}+1\right)\ge 2\sqrt{\alpha_i}+1\,.$$

Consider now the infinite Z-aggregate Z_0 and its boundary B_0. It is immediate that $\#B_0=4L+4$. The finite grid of node set N is a hole in N^{∞}, denoted H_{01}, circumscribed by $C_{01}=B_0$, and, for

every $1 \le j \le h$, $Z_j \subseteq H_{01}$. Then every $u_{xy} \in C_{01}$ is assigned the weight $\bar{w}_{xy}$, which accounts for the circumstance that u_{xy} may also be in the circumscribing set of some $Z_j \subseteq H_{01}$.

Let p_{xy}^{01} be the unique local pattern of u_{xy} in C_{01}, and $\bar{w}_{xy}^{01} = \frac{n_{xy}^{01}}{n_{xy}^{01} + m_{xy}^{01}}$. Beside Z_{01}, unit $u_{xy} \in C_{01}$ may be adjacent to Z-aggregates $Z_{02} \subseteq H_{01}, \ldots, Z_{0k} \subseteq H_{01}$, where $k \le n_{xy}^{01} + 1$, because among units in H_{01} being adjacent to u_{xy}, at most n_{xy}^{01} might belong to distinct aggregates Z_{ij}, $2 \le j \le k$.

Since unit u_{xy} contributes to $\bigcup_{i=0,h} B_i$ as a member of C_{01}, C_{02}, …, C_{0k}, and such contributions are evaluated independently, we need to ensure that the sum of the contributions of u_{xy} does not exceed 1. To this purpose, consider $Z_{01}, Z_{02}, \ldots, Z_{0k}$ and the respective $C_{01}, \ldots, C_{0k}$. The contribution of u_{xy} to C_{0j} ($1 \le j \le k$) is $w_{xy}^{0j} = \frac{n_{xy}^{0j}}{n_{xy}^{0j} + m_{xy}^{0j}}$. We claim that $\bar{w}_{xy}^{01} + \sum_{j=2,k} w_{xy}^{0j} \le 1$. In fact, for every j,l, $2 \le j,l \le k$, $j \ne l$ must be $N_{xy}^{0j} \subseteq N_{xy}^{01}$ and $N_{xy}^{0l} \subseteq M_{xy}^{0j}$, thus

$m_{xy}^{01} + \sum_{j=2,k} n_{xy}^{0j} \le m_{xy}^{01} + n_{xy}^{01}$ and $m_{xy}^{01} + \sum_{j=2,k} n_{xy}^{0j} \le m_{xy}^{0t} + n_{xy}^{0t}$ for any $t=2,\ldots,k$. Among the local patterns $p_{xy}^{01}, \ldots, p_{xy}^{0k}$ of u_{xy} in $C_{01}, \ldots, C_{0k}$, consider pattern p_{xy}^{0t} ($1 \le t \le k$) for which $n_{xy}^{0t} + m_{xy}^{0t}$ is minimum. We thus have $\frac{m_{xy}^{01}}{n_{xy}^{01} + m_{xy}^{01}} + \sum_{j=2,k} \frac{n_{xy}^{0j}}{n_{xy}^{0j} + m_{xy}^{0j}} \le \frac{m_{xy}^{01} + \sum_{j=2,k} n_{xy}^{0j}}{n_{xy}^{0t} + m_{xy}^{0t}} \le 1$.

The weight $\bar{w}_{xy}$ of every $u_{xy} \in C_{01}$ is easily determined by inspection, and the sum $\bar{w}_{01}$ of the weights of units in C_{01} is $\bar{w}_{01} = \frac{1}{2} 4L + \frac{3}{4} \cdot 4 = 2L + 3$. In conclusion, we obtain $\#\bigcup_{i=0,h}(C_i \cup \bigcup_{p=1,q} C_{ip}) \ge \sum_{i=1,h} \left(2\sqrt{\alpha_i} + 1\right) + 2L + 3$, which proves the lemma. ❑

Lower bounds to $\#\bigcup_{i=0,h} B_i$ holding for hexagonal, square and triangular grids are provided by the following lemmas:

Lemma 4. *In the infinite grid G_6^{∞}, let Z_0, Z_1, …, Z_h be the Z-aggregates defined by DAGS, B_i*

be the boundary of Z_i and α_i be the area of Z_i for i=1,…,h. If #N_f<n/2, then:

$$\#\bigcup_{i=0,h} B_i \geq \tfrac{1}{2}\sum_{i=1,h}\left(\sqrt{12\alpha_i - 3}+1\right)+2L+2.$$

Proof (sketch). The proof [16] proceeds by reasoning similar to those used in the proof of Lemma 3, based on the following preliminary results:

- the units in the circumscribing set C of any finite aggregate in G_6^{∞} are traversed by a simple circuit in G_6^{∞}. This property is derived by reasoning similar to those used in lemmas 3.1 and 3.2, but, due to the planarity of G_6^{∞}, there is no need of "pruning" the grid as it is done in Lemma 3.1.
- units in C are associated with unique local patterns, grouped into three classes with different weights and angles.
- Given any finite aggregate A of area α, the weights w and $\overline{w}$ of its circumscribing set are given by $w \geq \frac{1}{2}\left(\sqrt{12\alpha-3}+1\right)$ and $\overline{w} \geq \frac{1}{2}\left(\sqrt{12\alpha-3}+5\right)$.
- Any bridge is shared by at most two aggregates and the number γ of bridges among aggregates in the family H_{i0}, …, H_{iq} is at most $2q$. ❑

Lemma 5. *In the infinite grid G_4^{∞}, let Z_0, Z_1, …, Z_h be the Z-aggregates defined by DAGS, B_i be the boundary of Z_i and α_i be the area of Z_i for i=1,…,h. If #N_f<n/2, then:*

$$\#\bigcup_{i=0,h} B_i \geq \sum_{i=1,h}\left(\tfrac{17}{18}\sqrt{2\alpha_i - 1}-\tfrac{1}{6}\right)+2L+\tfrac{4}{3}.$$

Proof (sketch). The proof [6] proceeds by reasoning similar to those used in the proof of Lemma 3, based on the following preliminary results:

- the units in the circumscribing set C of any finite aggregate in G_4^{∞} are traversed by a simple circuit in the superimposed 8-grid G_8^{∞}. Due to the planarity of square grids,

the proof resembles the one used for hexagonal grids.

- units in C are associated with unique local patterns, grouped into five classes with different weights and angles.
- Given any finite aggregate A of area α, the weights w and $\overline{w}$ of its circumscribing set are given by $w \geq \frac{17}{18}\left(\sqrt{2\alpha-1}-\frac{1}{6}\right)$ and $\overline{w} \geq \frac{17}{18}\left(\sqrt{2\alpha-1}+\frac{37}{18}\right)$.
- Any bridge is shared by two or three aggregates and the number γ of bridges among aggregates in the family $H_{i0}, \ldots, H_{iq}$ is given by $\gamma \leq 2q-\tau$, where τ is the number of bridges shared by three aggregates. ❑

Lemma 6. *In the infinite grid G_3^{∞}, let $Z_0, Z_1, \ldots, Z_h$ be the Z-aggregates defined by DAGS, B_i be the boundary of Z_i and α_i be the area of Z_i for i=1,…,h. If $\#N_f<n/2$, then:*

$$\#\bigcup_{i=0,h} B_i \geq \sum_{i=1,h}\sqrt{\tfrac{2}{3}\alpha} + L + 2 .$$

Proof (sketch). The proof [3] is simpler than in the preceding cases since, at worst, every unit in the circumscribing set C is shared by exactly three aggregates and has weights w and $\overline{w}$ equal to 1/3. Then there is no circuit to consider, nor local patterns or bridges to analyze. ❑

Recalling that $F \cup D \supseteq \bigcup_{i=0,h} B_i$, where F and D are the sets of faulty and dual units constructed in Step 1 of DAGS, lower bounds to the cardinality of set $F \cup D$ in grids of different degrees are given by the following lemma:

Lemma 7. *Consider sets F and D and Z-aggregates $Z_1,\ldots,Z_h$ defined in the infinite grid G_d^{∞} by steps 1 and 2 of the diagnosis algorithm and let $L=\sqrt{n}$. If α is the maximum of $\#Z_1,\ldots,\#Z_h$, then:*

- *for d*=8, $\#(F \cup D) \geq \left(L^2+2L+1\right)\dfrac{2\sqrt{\alpha}+1}{\alpha+2\sqrt{\alpha}+1}+2L+3$;

– *for d*=6, $\#(F \cup D) \geq \left(L^2 + 2L\right)\frac{\frac{1}{2}\left(\sqrt{12\alpha - 3} + 1\right)}{\alpha + \frac{1}{2}\left(\sqrt{12\alpha - 3} + 1\right)} + 2L + 2$;

– *for d*=4, $\#(F \cup D) \geq \left(L^2 + 2L - \frac{4}{3}\right)\frac{17/18 \cdot \sqrt{2\alpha - 1} - 1/6}{\alpha + 17/18 \cdot \sqrt{2\alpha - 1} - 1/6} + 2L + \frac{4}{3}$;

– *for d*=3, $\#(F \cup D) \geq \left(L^2 + 2L - 2\right)\frac{\sqrt{\frac{2}{3}\alpha}}{\alpha + \sqrt{\frac{2}{3}\alpha}} + L + 2$.

Proof. Consider the case of *d*=8. The proofs holding for the remaining cases are derived straightforwardly along the same guidelines.

By construction, $L^2 + 4L + 4 = \sum_{i=1,h}\alpha_i + \#(F \cup D)$, where $\alpha_i = \#Z_i$ for *i*=1, …,*h*. From Lemma 3, it is immediate that $\#(F \cup D) = \sum_{i=1,h}\left(2\sqrt{\alpha_i} + 1\right) + 2L + 3 + r$, with *r*≥0.

Letting $f(\alpha_i) = \frac{2\sqrt{\alpha_i} + 1}{\alpha_i + 2\sqrt{\alpha_i} + 1}$, we obtain: $\#(F \cup D) = \sum_{i=1,h} f(\alpha_i)\left(\alpha_i + 2\sqrt{\alpha_i} + 1\right) + 2L + 3 + r$.

Observing that $f(\alpha_i)$ is a decreasing function of α_i when $\alpha_i \geq 1$, from $1 \leq \alpha_i \leq \alpha$ we have:

$$\#(F \cup D) \geq f(\alpha)\left(\sum_{i=1,h}\alpha_i + \sum_{i=1,h}(2\sqrt{\alpha_i} + 1)\right) + 2L + 3 + r =$$

$$= f(\alpha)\left(\sum_{i=1,h}\alpha_i + \#(F \cup D) - r - 2L - 3\right) + 2L + 3 + r =$$

$$= f(\alpha)\left(L^2 + 4L + 4 - r - 2L - 3\right) + 2L + 3 + r = \left(L^2 + 2L + 1\right) f(\alpha) + r(1 - f(\alpha)) + 2L + 3$$

Observing that $f(\alpha) < 1$, we conclude $\#(F \cup D) \geq \left(L^2 + 2L + 1\right)\frac{2\sqrt{\alpha} + 1}{\alpha + 2\sqrt{\alpha} + 1} + 2L + 3$. ❑

We are now ready to provide a lower bound to $\#N_f$, which is independent of the number of aggregates and of their individual areas and shapes.

Theorem 2. *Consider grid* G_d *of size* $L = \sqrt{n}$ *and assume* $\#N_f < n/2$*., The Fault-Free Core is*

entirely fault-free for any admissible syndrome of $\mathbf{N}_f$ *if* $\#\mathbf{N}_f < T_d$*, with:*

– $T_8 = \min\limits_{1\le\alpha\le n}\left(\left(n+2\sqrt{n}+1\right)\dfrac{2\sqrt{\alpha}+1}{2\left(\alpha+2\sqrt{\alpha}+1\right)}-\left(\sqrt{n}+\dfrac{1}{2}\right)+\alpha\right)$;

– $T_6 = \min\limits_{1\le\alpha\le n}\left(\frac{1}{2}\left(n+2\sqrt{n}\right)\dfrac{\sqrt{12\alpha-3}+1}{2\alpha+\sqrt{12\alpha-3}+1}-\sqrt{n}+\alpha\right)$;

– $T_4 = \min\limits_{1\le\alpha\le n}\left(\left(n+2\sqrt{n}-\frac{4}{3}\right)\dfrac{17/18\cdot\sqrt{2\alpha-1}-1/6}{2\left(\alpha+17/18\cdot\sqrt{2\alpha-1}-1/6\right)}+\alpha-\sqrt{n}+\frac{2}{3}\right)$;

– $T_3 = \min\limits_{1\le\alpha\le n}\left(\dfrac{n+2\sqrt{n}-2}{\sqrt{6\alpha}+2}-2\sqrt{n}+2+\alpha\right)$.

Proof. Consider the case of d=8. The proofs in the remaining cases are derived straightforwardly along the same guidelines.

Under the hypothesis $\#\mathbf{N}_f < n/2$, there exists at least one Z-aggregate other than Z_0. Consider sets $\mathbf{F}$ and $\mathbf{D}$ and Z-aggregates defined by steps 1 and 2 of the diagnosis algorithm in the infinite grid G_8^{∞}, and let α be the maximum of the Z-aggregate cardinalities. Since $4L$+4 units in set $\mathbf{F}$ are dummy and $\mathbf{D}\subseteq\mathbf{N}$, we obtain $\#((\mathbf{F}\cup\mathbf{D})\cap\mathbf{N}) = \#(\mathbf{F}\cup\mathbf{D})-4L-4$. From Lemma 7:

$$\#(\mathbf{F}\cup\mathbf{D}) \ge \left(L^2+2L+1\right)\frac{2\sqrt{\alpha}+1}{\alpha+2\sqrt{\alpha}+1}+2L+3 \Rightarrow$$

$$\#((\mathbf{F}\cup\mathbf{D})\cap\mathbf{N}) \ge \left(L^2+2L+1\right)\frac{2\sqrt{\alpha}+1}{\alpha+2\sqrt{\alpha}+1}-2L-1$$

Assume that any $\mathbf{Z}_i$, which is a member of the $\mathbf{FFC}$, is not fault-free and thus completely faulty. Recalling that at least half of the units in $(\mathbf{F}\cup\mathbf{D})\cap\mathbf{N}$ must be faulty, and $\#\mathbf{Z}_i=\alpha$, the number of faults in $\mathbf{N}$ is at least $T_8(\alpha)$, with $T_8(\alpha)=\left(n+2\sqrt{n}+1\right)\dfrac{2\sqrt{\alpha}+1}{2\left(\alpha+2\sqrt{\alpha}+1\right)}-\sqrt{n}+\frac{1}{2}+\alpha$.

However, if the actual number of faults is less than $T_8(\alpha)$, Z_i must be fault-free. Since Z_i was chosen as an arbitrary member of the *FFC*, all units in the *FFC* must be non-faulty.

The parameter α is dependent on the syndrome. The bound holding for every possible syndrome is derived by letting $T_8 = \min_{1 \le \alpha \le n} T_8(\alpha)$. ❑

Table 4. Values of the syndrome-independent bound T_d for triangular, square, hexagonal and octagonal grids of different sizes.

Function T_d cannot be expressed analytically, because the equation yielding the minimum cannot be solved algebraically. However, bound T_d holding for grids of different degrees and sizes is tightly upper and lower bounded by Theorem 3. A numerical evaluation of T_d is listed in Table 4. Observe that, for any given size, bound T_d increases as the degree of the grid increases, thus confirming the same behavior of the average syndrome-dependent bound T_σ, which resulted from simulation experiments [5, 6, 16].

Tight bounds to function T_d are provided analytically by the following theorem.

Theorem 3. *Bound T_d is limited from below and from above by the following expressions:*

- *if d=8 and n≥16:*

$$\left(\tfrac{27}{4}\right)^{\frac{1}{3}}\left(n+2\sqrt{n}+1\right)^{\frac{2}{3}} - \tfrac{16}{3}\left(\tfrac{3}{4}\left(n+2\sqrt{n}+1\right)\right)^{\frac{1}{3}} - \sqrt{n} + \tfrac{23}{18} < T_8 < \left(\tfrac{27}{4}\right)^{\frac{1}{3}}\left(n+2\sqrt{n}+1\right)^{\frac{2}{3}} - \sqrt{n} - \tfrac{1}{2}$$

- *if d=6 and n≥30:*

$$\left(\tfrac{9}{4}\left(n+2\sqrt{n}\right)\right)^{\frac{2}{3}} - 4\left(n+2\sqrt{n}\right)^{\frac{1}{3}} + \tfrac{35}{12} - \sqrt{n} < T_6 < \left(\tfrac{9}{4}\left(n+2\sqrt{n}\right)\right)^{\frac{2}{3}} - \sqrt{n} + \tfrac{1}{4}$$

- *if d=4 and n≥9:*

$$\tfrac{3}{2}\left(\tfrac{17}{18}\left(n+2\sqrt{n}-\tfrac{4}{3}\right)\right)^{\frac{2}{3}} - \tfrac{170}{27}\left(\tfrac{10}{9}\left(n+2\sqrt{n}-\tfrac{4}{3}\right)\right)^{\frac{1}{3}} - \sqrt{n} + \tfrac{23}{3} \le T_4 \le \tfrac{3}{2}\left(\tfrac{17}{18}\left(n+2\sqrt{n}-\tfrac{4}{3}\right)\right)^{\frac{2}{3}} - \sqrt{n} + \tfrac{7}{6}$$

- *if d=3 and n≥4:*

$$\frac{1}{2}3^{\frac{2}{3}}\left(n+2\sqrt{n}-2\right)^{\frac{2}{3}}-\frac{1}{4}3^{\frac{1}{3}}\left(n+2\sqrt{n}-2\right)^{\frac{1}{3}}-2\sqrt{n}+\frac{62}{27}\le T_3\le\frac{1}{2}3^{\frac{2}{3}}\left(n+2\sqrt{n}-2\right)^{\frac{2}{3}}-2L+2\,.$$

Proof. Consider the case of d=8. The proofs for the remaining cases are similar.

The expression of $T_8(\alpha)$ can be rearranged as:

$$T_8(\alpha)=\min_{1\le\alpha\le n}\left(\left(n+2\sqrt{n}+1\right)\frac{2\sqrt{\alpha}+1}{2\left(\alpha+2\sqrt{\alpha}+1\right)}-\left(\sqrt{n}+\tfrac{1}{2}\right)+\alpha\right).$$

Letting $h(\alpha)=\frac{M}{\sqrt{\alpha}+2}+\alpha-L-\frac{1}{2}$ with $M=n+2\sqrt{n}+1$, it is seen that $T_8(\alpha)>h(\alpha)$ for $\alpha\ge1$. The minimum $h=\min_{1\le\alpha\le n}h(\alpha)$, which is determined by solving the equation $\frac{\partial h(\alpha)}{\partial\alpha}=0$, is given by:

$$h=\frac{M}{\sqrt[3]{\frac{16}{54}+\frac{M}{4}+\sqrt{\frac{M^2}{2^4}+\frac{2^2M}{3^3}}}+\sqrt[3]{\frac{16}{54}+\frac{M}{4}-\sqrt{\frac{M^2}{2^4}+\frac{2^2M}{3^3}}}-\frac{4}{3}+2}+\left(\sqrt[3]{\frac{16}{54}+\frac{M}{4}+\sqrt{\frac{M^2}{2^4}+\frac{2^2M}{3^3}}}+\sqrt[3]{\frac{16}{54}+\frac{M}{4}-\sqrt{\frac{M^2}{2^4}+\frac{2^2M}{3^3}}}-\frac{4}{3}\right)^2-L-\frac{1}{2}$$

Observing that, for $n\ge16$:

$$h>\frac{M}{\sqrt[3]{2\frac{2^3}{3^3}+\frac{M}{2}+\frac{4}{3}}}+\left(\sqrt[3]{\frac{2^3}{3^3}+\frac{M}{2}}-\frac{4}{3}\right)^2-L-\frac{1}{2}>\left(\frac{27}{4}\right)^{\frac{1}{3}}M^{\frac{2}{3}}-\frac{16}{3}\left(\frac{3}{4}M\right)^{\frac{1}{3}}-L+\frac{23}{18}$$

we obtain $T_8>h$, with $h>\left(\frac{27}{4}\right)^{\frac{1}{3}}\left(n+2\sqrt{n}+1\right)^{\frac{2}{3}}-\frac{16}{3}\left(\frac{3}{4}\left(n+2\sqrt{n}+1\right)\right)^{\frac{1}{3}}-\sqrt{n}+\frac{23}{18}$.

Similarly, letting $g(\alpha)=\frac{n+2\sqrt{n}+1}{\sqrt{\alpha}}-\sqrt{n}-\frac{1}{2}+\alpha$, it is seen that $T(\alpha)<g(\alpha)$ for $\alpha\ge1$. The minimum $g=\min_{1\le\alpha\le n}g(\alpha)$ is determined as $g=\left(\frac{27}{4}\right)^{\frac{1}{3}}\left(n+2\sqrt{n}+1\right)^{\frac{2}{3}}-\sqrt{n}-\frac{1}{2}$.

The thesis follows by combining $T_8<g$ and $T_8>h$. ❑

The following corollary is immediate from Theorem 3.

Corollary 1. *In grids of degrees 3,4,6 and 8, bound T_d is $\Theta(n^{2/3})$.*

5 Conclusions

A diagnosis algorithm aimed at providing correct diagnosis for two-dimensional grids of

different degrees has been presented. The algorithm, named DAGS (Diagnosis Algorithm for Grid Structures), operates in three steps. The first step performs a preliminary classification of units based on the outcomes of reciprocal tests. The second step identifies a Fault-Free Core and returns a syndrome-dependent bound T_σ, with the property that all the units in the Fault-Free Core are actually non-faulty if the cardinality of the actual fault set is below T_σ. The last step exploits the reliable tests executed by the units in the Fault-Free Core to identify the state of more units, both faulty and/or non-faulty.

The syndrome-independent bound T_d ensuring correctness independently of the actual syndrome is derived by a worst-case analysis covering the cases of octagonal, hexagonal, square and triangular grids. Essentially, the bound is determined as the minimum of the syndrome-dependent bound T_σ over the set of all syndromes that give rise to at least one Z-aggregate. This property is ensured if the actual number of faults is below $n/2$. The technique used to this purpose is based on a geometrical analysis of the Z-aggregates. The bound T_d is evaluated both analytically and numerically. Although T_d is $\Theta(n^{2/3})$ independently of the degree d of the grid, it is shown that its actual value is considerably influenced by the value of d.

6 Appendix

This appendix contains the proof to Lemma 2 introduced in Section 4, together with some preliminary definitions and propositions. The proof is based on a geometrical analysis of the aggregates and of their boundaries. For the ease of presentation, we limit consideration to octagonal grids.

Let $a,b,c,d \geq 0$ be integers, with $b>a+1$ and $d>c+1$; we define the *rectangle* of parameters a,b,c,d as the aggregate $R(a,b,c,d) = \{u_{xy} \in N^\infty : a<x<b,\ c<y<d\}$. Rectangle $R(1,5,1,5)$ is shown in Figure 11 *a*.

The following lemma characterizes the perimeter and area of rectangles of given parameters.

Lemma A1. *Let R=R(a,b,c,d) be an arbitrary rectangle and let* B_R *be its boundary; the perimeter and area of R are given by* $\#B_R = 2\cdot(b-a+d-c)$ *and* $\#R=(b-a-1)(d-c-1)$*, respectively.*

Proof. Observe that $\#B_R$ equals the perimeter of the polygon $P(a,b,c,d)=\{(x, y)\in \Re^2: a<x<b, c<y<d\}$, which is $2\cdot((b-a)+(d-c))$.

The area of R may be easily computed observing that it is composed by $(b-a-1)$ lines of $(d-c-1)$ units each, then $\#R=(b-a-1)(d-c-1)$. ❑

A rectangle $R=R(a,b,c,d)$ with perimeter $p=\#B_R$ is said to be *maximal* for perimeter p if $\#R\geq\#R'$ for every other rectangle R' of perimeter p. The maximal rectangle maximizes the area for a given perimeter. Lemma A2 characterizes the class of maximal rectangles, providing a lower bound to the perimeter of maximal rectangles of given area.

Lemma A2. *Let R=R(a,b,c,d) be any maximal rectangle and p its perimeter. Then* $p\geq 4+4\sqrt{\#R}$.

Proof. The perimeter of R, is $p = 2\cdot(b-a+d-c) = 2\cdot(b-a-1)+2\cdot(d-c-1)+4$. Letting $z=(b-a-1)$, we have $p =2z+2\cdot(d-c-1)+4$, from which we derive:

$$(d-c-1)= p/2 - z - 2. \qquad (4)$$

By substituting (4) in the expression of the area of R we obtain $\#R = z\cdot(p/2-z-2)$. This expression relates the area of the rectangle to its perimeter. Given a perimeter p, the value of z that maximizes the area of R is $z=p/4-1$, and the rectangle of maximum area is a square with $\#R=(p/4-1)^2$. Solving the last equation with respect to p, it is concluded that $p= 4+4\sqrt{\#R}$.

Observe that the value of p obtained by maximizing $\#R$ may not be integer and, even if integer, may not correspond to any square (to correspond to a square it must be p mod 4=0).

However, maximal rectangles must have a perimeter greater than or equal to the one we obtained by maximizing $\#R$, hence $p\geq 4+4\sqrt{\#R}$. ❑

The rectangle $R(1,5,1,5)$ shown in Figure 11 *a* is maximal.

The preceding results enable determining a lower bound to the cardinality of the circumscribing set of any aggregate in G_8^{∞}, as stated by the following lemma, which corresponds to the proposition of Lemma 2 covering the case of octagonal grids.

Lemma A3. *For any finite aggregate A of area* α *in* G_8^{∞}*, the cardinality* $\#C$ *of the circumscribing set* C *of A is* $\#C \geq 4\sqrt{\alpha} + 4$.

Proof. Given any finite aggregate A in G_8^{∞} and its circumscribing set C, we define parameters a,b,c,d as follows: a= min{x: $u_{xy} \in C$}; b= max{x: $u_{xy} \in C$}; c= min{y: $u_{xy} \in C$}; d= max{y: $u_{xy} \in C$}

These parameters define rectangle $R=R(a,b,c,d)$. Observe that the boundary B_R of R circumscribes aggregate A (see Figure 11 b); moreover, it can be easily seen that $\#B_R \leq \#C$ and $\#R \geq \alpha$.

If R is a maximal rectangle, then $\#C \geq \#B_R \geq 4+4\sqrt{\#R} \geq 4+4\sqrt{\alpha}$.

Otherwise, let R' be the maximal rectangle of perimeter $\#B_R$; we have $\#R' \geq \#R \geq \alpha$, then $\#C \geq \#B_R \geq 4+4\sqrt{\#R'} \geq 4+4\sqrt{\#R} \geq 4+4\sqrt{\alpha}$. ❑

Figure 11: a) Rectangle $R(1,5,1,5)$; b) Aggregate A and circumscribing boundary B_R.

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Table 1: Invalidation rule in the PMC model.

Testing unit	Tested unit	Test outcome
Fault-free	Fault-free	0
Fault-free	Faulty	1
Faulty	Fault-free	0 or 1
Faulty	Faulty	0 or 1

Table 2. The DAGS algorithm.

```
{initialization: sets F, D, and Z are empty}
   F:= ∅; D:= ∅; Z:= ∅;
{Step one: Local Diagnosis}
   {identification of faulty units}
   While ∃u∈N−F, ∃v∈N: u ←(1 0)→ v Do
         F:=F∪{u} {u is faulty by Property (a)}
   EndWhile
   F:=F∪A0(F) {F is augmented according to Property (b)}
   {identification of dual units}
   While ∃u, v∈N−(F∪ D): u ←(1 1)→ v Do
         D:=D∪{u, v}
   EndWhile
   Z:=N−(D∪F); {identification of set Z of zero-units}
{Step Two: Fault-Free Core Identification}
   Q:=EmptyQueue; h:=1; {Z-aggregate count}
   While ∃u∈Z Do
         {u is removed from Z and inserted into Q}
         Z:=Z−{u}; h:=h+1; Zh:={u}; insert(u,Q);
         While NonEmpty(Q) Do
               u:=remove(Q);
               For every u∈Z: u↔v Do
                  Z:=Z−{v}; Zh:=Zh∪{v};
                  insert(v,Q)
               EndFor
         EndWhile
   EndWhile;
   α:=MaxCard(Z1,...,Zh);{given sets S1,...,Sn MaxCard(S1,...,Sn) returns
                          the maximum of #S1,...,#Sn}
   FFC:=⋃#Zi=α Zi {set FFC is the Fault-Free Core}
   Tσ:= α +#F+#D/2;   {syndrome-dependent bound}
Assertion: the diagnosis is correct if α>0 and #Nf<Tσ;
{Step Three: Augmentation}
   FFC:=FFC∪D0(FFC); {the zero descendants of the FFC are non faulty}
   F:=F∪Δ1(FFC); {the units in the disagreement set of the FFC are faulty}
   F:=F∪A0(F); {zero ancestors of faulty units are faulty}
{Diagnosis and Validation}
S:=N−(FFC∪F); {S is the set of suspect units}
return(FFC, F, S, Tσ);
```

Table 3. Classification of local patterns for octagonal grids.

class	w_{xy}	β_{xy}
1	1/4	$-\pi/2$
2	3/4	$\pi/2$
3	1/2	0

Table 4. Values of the syndrome-independent bound T_d for triangular, square, hexagonal and octagonal grids of different sizes.

n	G_3	G_4	G_6	G_8
256	9	34	46	52
1024	38	101	129	145
4096	130	281	351	392
16384	399	762	937	1044
65536	1150	2022	2463	2737

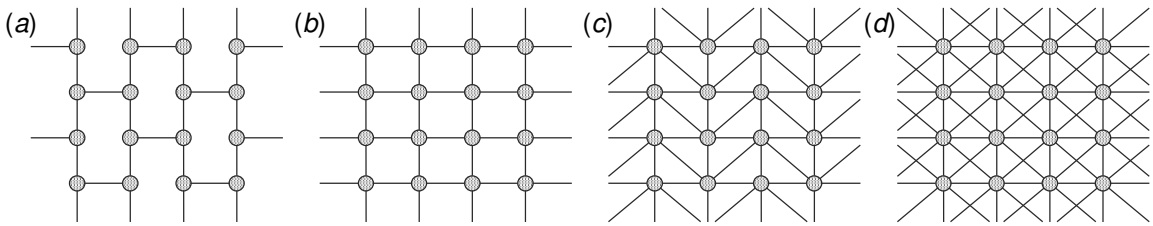

Figure 1: Grids G_3^T (*a*), G_4^T (*b*), G_6^T (*c*), and G_8^T (*d*) of size 16.

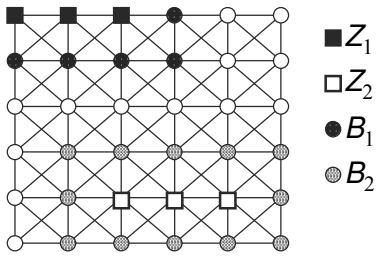

Figure 2: Two isomorphic Z-aggregates with different perimeters.

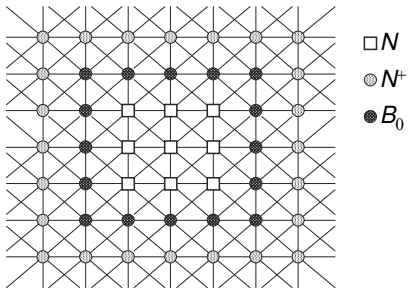

Figure 3: The infinite grid G_8^{∞} .

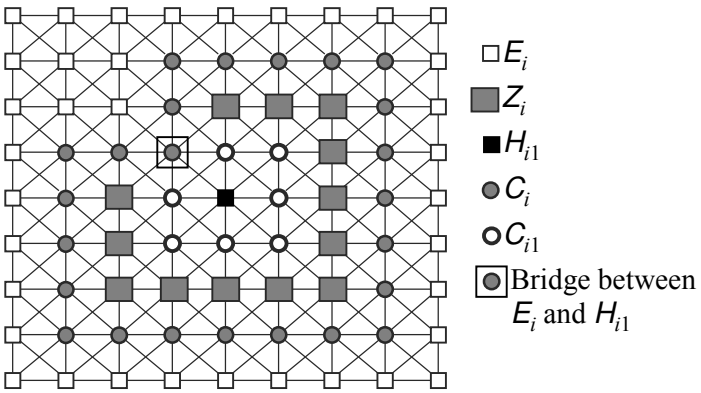

Figure 4: Z-Aggregate Z_i ($i \geq 1$).

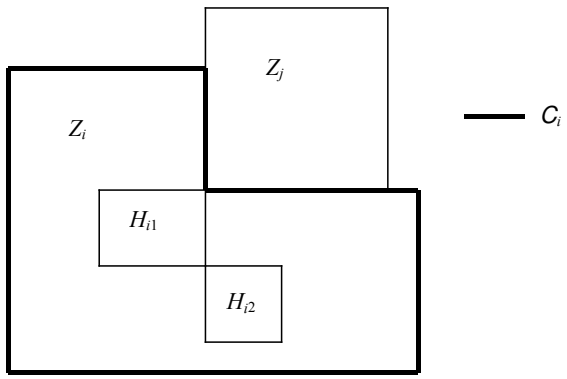

Figure 5: Z-aggregates, holes and bridges.

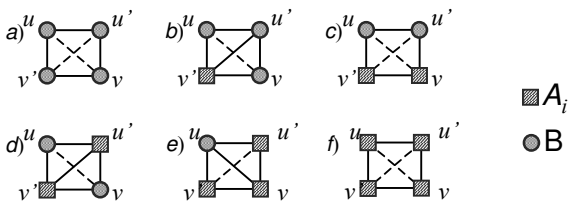

Figure 6: Pruning of X: dotted lines denote the edges which are removed.

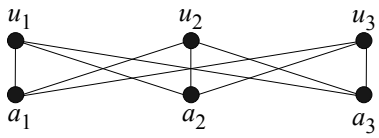

Figure 7: Graph deriving from contracting A_1, A_2 and A_3 in graph X^p.

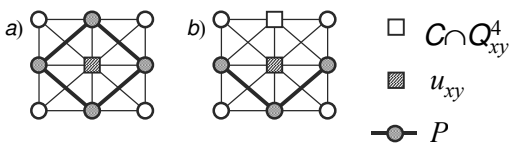

Figure 8: Path in Q^8_{xy} for k=0 (*a*)) and k=1 (*b*)).

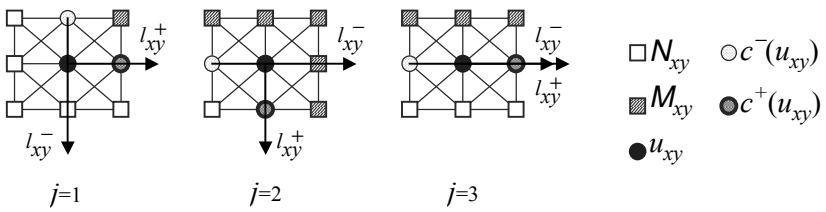

Figure 9: Local patterns.

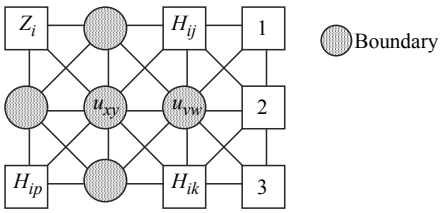

Figure 10: Z-aggregate Z_i and holes H_{ip}, H_{ik} and H_{ij}.

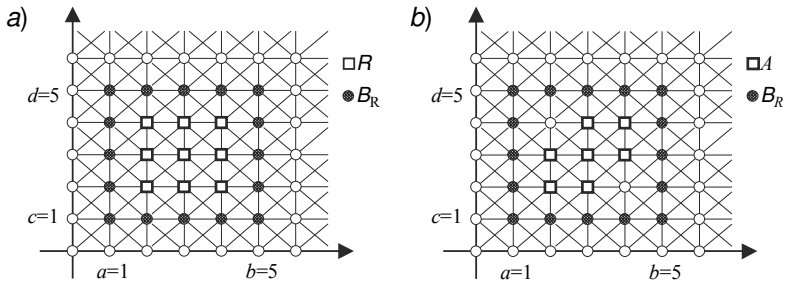

Figure 11: *a*) Rectangle $R(1,5,1,5)$; *b*) Aggregate A and circumscribing boundary B_R.