Chapter 1

Rendering Equation Revisited: how to Avoid Explicit Visibility Computations

Marco Pellegrini*

Abstract

The rendering integral equation was introduced by Kajiya [16] to model the equilibrium of radiant energy exchange in a 3D scene with non partecipating medium and surfaces exhibiting a variety of local illumination effects including diffuse, glossy and specular reflectance. This equation has been extensively applied in the area of photorealistic rendering, specially in its radiosity form. In this paper we first derive a new form of such an integral equation and then we apply to the new equation the radiosity solution technique. There are two main novelties with respect to the previously known radiosity treatments. First of all we avoid any explicit global visibility computation by simulating visibility as a superimposition of independent effects at the level of the integral equation. The calculations do not need to carry around any explicit visibility information; nor such visibility information is kept in expensive auxiliary data structures. Secondly, by use of integral geometric transformations, we reduce the six-dimensional integrals representing coefficients of the radiosity matrix to two-dimensional ones. We show that such integrals can then be reduced to a form for which a result in [25] gives an high precision numerical approximation algorithm. The technique presented works for any piecewise polynomial functional basis adopted in generalized radiosity discretization and places mild conditions on the other functions of the model.

1 Introduction

1.1 Handling occlusions

In the first part of the paper we give a new form of the integral rendering equation whose main feature is that occlusions are simulated by superimposition. The intuitive idea is that we allow light energy to go through opaque objects but, when this happens, we generate an additional flow of energy in the same direction of equal magnitude but with opposite sign, so that positive and negative energies cancel out behind the object to give the occlusion effect. This change of sign is obtained by extending the domain of the *bidirectional reflectance distribution function* (BRDF) inside the objects and is thus a purely local feature of the model. This approach results in avoidance of the use of any auxiliary data structure to keep track of visibility or any other explicit visibility calculation. Instead occlusions are treated within the same formalism (and therefore the same code) we would use to describe unoccluded rendering factors. Thus we have reduced the problem of computing rendering factor with occlusions to that of rendering factors without occlusions. The superimposition itself is attained implicitly during the solution of the linear system.

The handling of occlusion by superimposition is exact at the level of the integral equation (that is, the solution of the integral equation is unchanged) and not heuristic as the methods mentioned in 1.5. In other words the discretization approximation introduced are just those inherent to many numerical schemes, such as those for electrostatics, where superimposition is exploited.

Willmot and Heckbert [43] point out that poor treatment of occlusions in known wavelet methods is a source of poor quality in some test images. The occlusion handling technique shown in this paper, being mathematically sound, has a fair chance of eliminating such source of problems. However it should be noted that the treatment shown in this paper is independent of any particular choice of basis function so its main features, (i.e. handling of occlusions and integration technique) can be used with any basis (wavelet or not) which are known to give further benefits.

1.2 Evaluation of rendering factors

In the second part of the paper we give a new discrete form of the rendering equation based on integral geometric transformations. Such integral geometric tricks have been applied to radiosity equations in [23] and to electrostatic problems in [22]. The main advantage with respect to the the previous derivations is that the entries of the matrix are represented by two-dimensional integrals, instead of six-dimensional integrals. Thus much fewer kernel evaluations are needed in general to approximate each coefficient to the same precision. With some mild hypotheses on the analytic

*Istituto di Matematica Computazionale del CNR, Via S.Maria 46, 56126-Pisa (Italy). e-mail: pellegrini@imc.pi.cnr.it

form of the basis and of the BDRF such coefficients can be reduced to a form very similar to that of higher order form factors arising in the discretization of radiosity equations. This similarity implies that we can apply an integration algorithm in [25] which gives an absolute error ϵ in time $O(\log_c^2(1/\epsilon))$, for a real $c > 1$, on each entry of the rendering matrix. The real value c does not depend on ϵ; it depends however on the input data.

1.3 Background: The Rendering Equation

The rendering equation has been introduced by Kajiya [16] to describe the equilibrium condition of radiative energy exchange among a set of surfaces within a non-partecipating medium in 3-space. It is customary to distinguish between local light effects and global light effects. Local light effects (such as diffuse, specular and glossy reflectance of materials) are modelled by the *bidirectional reflectance distribution function* (BRDF for short) which represents the amount of energy received at a point of a surface from some direction and re-emitted in a second direction [33, 9]. Global illumination captures the effects due to light exchanged by surfaces and is usually based on the integral equation of Kajiya [16], the *rendering equation*.

The *radiosity* equation is derived from the rendering equations with the additional hypothesis that the BRDF does not depend on the directions of incoming and outgoing light. This corresponds to modelling only diffuse reflection, while mirror and glossy effects are lost.

Even for the simplest geometries no exact solution is known and therefore numerical approximation methods are used to solve such equations. The main approaches to the solution of rendering (and radiosity) equations can be classified as follows:

- Ray tracing approach (see [4, 42] and referenced therein).
- Radiosity or Boundary Elements approach (see [33, 9] and references therein). For a general introduction to finite element methods in radiosity we refer to [13], [14] and [5].
- Mixed methods (see [19] and references therein).

We place our research within the boundary element approach and in particular within the Galerkin-type radiosity methods (see [44, 41]). It is not our purpose to compare the relative merits of finite element, ray-shooting or mixed methods. The purpose of this paper is to address issues raised by computational bottlenecks in known boundary elements approaches. Three main computational bottlenecks in boundary element methods applied to rendering equations are the following:

- Handling of occlusions;
- High dimensional numerical integration;
- Density of the radiance matrix.

Below we give an overview of numerical methods applied to radiosity and rendering equations and a discussion of the three bottlenecks mentioned above.

1.4 Background: Numerical methods for rendering equation

At an high level the Galerkin method consists of two steps. In the first step the unknown function which is the solution of the integral equation is replaced by the linear combination of known functions taken from a finite set of functions called the *basis*. The elements of the basis are known, while the real coefficients of the linear combination are the variables to be determined. We can multiply both sides of any integral equation by any function (called a *test function*) defined on the same domain of the unknown function and then integrate over such a domain to obtain a new equation. The Galerkin method consists in taking as test functions exactly the basis. In the derived equations the only variables are the real coefficients of the linear combinations, therefore we can use linearity of integral operators to take them out of the integral sign. What is left is a linear equation where the coefficients are integrals involving the integral operator and the basis functions. Such coefficients do not seem to have a standard name and we shall refer to them as the *rendering factors*. For more details on finite element methods applied in radiosity and rendering we refer the reader to [13], [14] and [5].

Early work on extending numerical methods for the radiosity equations to rendering equations with non-diffuse reflection are in [15, 31, 32, 34]. In [6] the rendering equation is used as a starting point for the derivation of a discretized radiance system of linear equations. However the approach in [6] is not a pure Galerkin approach since we have (using a piecewise constant basis) a variable for each *pair of patches* from different polygons and a coefficient for each *triplet of patches* from three different polygons. Using an argument similar to one in [11] it is then justified the experimental finding that the number of significant coefficients actually scales linearly with the number of patches in the refinement, while such number is cubic in the number of input polygons. The derivation of the discretized linear system used in [10] as well as here is different from the one in [6] and closer to the standard Galerkin method; in particular (for the piecewise constant basis) only one variable per patch is needed. The use of wavelet basis within rendering has been explored by Gortler, Schröder, Cohen and Hanrahan [10, 29, 30].

Following [30] the rendering equations can be written in two different coordinate systems. In the first system, called *directional* parameterization, the radiance L is a function of a point on the surfaces of the scene and a direction (see (2.1) below). In the second system, called *spatial* parameterization the radiance L is function of pairs of points on the surfaces (see (2.2) below). A recent algorithm described in [30] proposes a numerical solution of such equation (in both formulations) by using a wavelet functional basis. Provided certain smoothness properties of the kernels are satisfied, the vanishing moments property associated with wavelets implies that the resulting matrix operator is sparse [7, 1]. Each nonzero element of the operator matrix is represented as a 6-dimensional integral which is evaluated in [30] using six-dimensional Gaussian product quadrature rules. For such integrals complete visibility is assumed among the supporting patches. Such visibility condition property is conservatively established and maintained under successive hierarchical refinements using an auxiliary data structure described in [38].

1.5 Previous methods for handling occlusions

Occlusions are well known source of theoretical and practical difficulties in rendering. Occlusions play a prominent role in hidden surface removal, in ray shooting and in any type of visibility problem. Here we concentrate on the role of occlusions in rendering computations within a boundary element type of approach.

One approach is to compute explicitly visibility maps, either in a continuous or a rasterized setting. In the well known hemicube method [8] occlusions in the computation of form factors are treated by using a z-buffering algorithm applied on the surface of the hemicube. A similar approach is used in [18]. Continuous visibility maps are computed by a sweeping approach and dynamic maintenance of planar maps in [21, 23, 37].

A second approach is to separate the rendering factor in the product of two terms, one is the rendering factor integral where the visibility function is neglected. The second term is a correcting factor that takes into consideration only the visibility either in a discrete or in a continuous setting. Hanrahan et al. [11] apply ray shooting operations to determine an occlusion-corrective parameter used as multiplier to the the result of the computation a form factor which disregard occlusions. This approach requires a ray shooting auxiliary data structure, such as binary space partition trees (BSP). Zatz [44] deals with the visibility function (see formula (2.3) below) within the calculation of the matrix coefficients by moving it out of the integration sign and integrating the visibility function separately from the residual (intuitively this corresponds to changing the integral of a product of functions with the product of the integrals of the functions). Naturally such operation gives a close approximation of the original integral only when the two functions have a low correlation. In general however it is hard to tell a priori whether such assumption is justified for the input at hand. More recently, in [36], a similar assumption is used to approximate the integrals defining the pointwise irradiance [33].

A third approach in [29, 30] consists in computing rendering factors only for fully visible pairs of patches, while for any other pair a default zero value is set as the corresponding entry of the radiance matrix. An auxiliary data structure described in [38] is invoked to store such visibility information and dynamically updated at each level of mesh refinement. This data structure requires non trivial computations in 5-dimensional Plücker space [24].

In certain contexts like a walkthrough in large in buildings [39], occlusions may be beneficial. Large occluders produce a partition of the visibility computation due to the fact that large walls completely occlude many pairs of patches. We can avoid a considering such pairs, thus improving the sparsity of the resulting matrix. In other scenes like a single room with many small objects, occlusions do not significantly reduce a priori the number of pairs to be considered,thus keeping track of pairs of completely visible, partially visible or completely occluded pairs of patches becomes a cumbersome side task which does not payoff by speeding up later stages of the computation.

1.6 On the sparsity of the rendering matrix

The next important bottleneck is the density/sparsity of the resulting rendering matrix. Let us suppose that we have as input a set of n interior disjoint triangles and that in the basis we use a k-terms expansion of the solution, where $k = n + m$. The rendering matrix will have $O((n+m)^2)$ terms. Several techniques have been explored to make the matrix sparse by avoiding the computation of coefficients that are zero or very small. For example, hierarchical radiosity/rendering [11] aims at reducing the dependency of the complexity in the parameter m. As noted in [30], results in [7, 1], imply that, provided some smoothness conditions are satisfied, a radiosity approach with a wavelet basis produces an operator matrix sparse in m (independently of additional sparsity due to occlusions). Clustering techniques, such as those in [35, 2], aim at reducing the dependency of the complexity on the parameter n.

In this paper we do not introduce any new sparsification technique, we notice however that our discussion is for a generic (piecewise polynomial) choice of a basis function, therefore it is compatible with wavelet-based sparsification techniques.

More generally, the techniques used in this paper are compatible and complementary with many other techniques (e.g. for meshing, for adaptive choice of functional basis) and the final quality of any prototype will depend also on the choice of such techniques aimed at obtaining additional features such as sparsity and/or continuity of the output solution. The discussion of such auxiliary methods although important in any implementation is nonetheless out of the scope of this paper.

1.7 Summary of results in this paper

The work presented in this paper has its roots is some ideas and techniques for computing form factors described in [21, 23].[1] The work presented in this paper differs significantly from [21, 23] in two aspects. First of all we deal with rendering equations whose (exact) solution is defined over a 4-dimensional domain, while the radiosity equation is defined over a 2-dimensional domain. This impacts on the dimension of the integrals (which is now 6) and we show how the dimension can be reduced to 2 by integrating analytically on 4 variables. The algorithm used for the numerical integration in [25] is novel with respect to [21, 23].

Secondly, we avoid computing any discrete or combinatorial visibility map or any other explicit visibility computation. We can summarize the main result of this paper in the following theorem:

THEOREM 1. *Given a set of n interior-disjoint triangles in 3-space, piecewise polynomial emission functions and piecewise polynomial BRDF, we can compute all entries of the rendering matrix corresponding to a k-terms expansion in a polynomial basis of the solution of (a modified form of) the rendering equation (with $k \geq n$), within absolute error ϵ per entry, in time $O(k^2 \log^2(1/\epsilon))$.*

Polynomials are in Cartesian coordinates and the set of triangles may (and in general will) present occlusions. The admissible BRDF include also the functions needed to simulate occlusions and specular effects which are based on the Dirac delta function. Triangles will be considered endowed with infinitesimal width so that we can define the notion of interior and exterior relative to R^3. When the triangles form boundaries of fully dimensional polyhedra (not necessarily convex) in 3-space the notion of interior/exterior can be associated with the polyhedra themselves.

1.8 Organization of the paper

The present paper is organized as follows. In Section 2 we derive new continuous and discrete form of the rendering equation. In Section 3 we discuss the transformation of 6-dimensional integrals into 2-dimensional ones and their numerical evaluation. In Section 4 we comment on the conditions imposed on the BDRF.

2 Continuous and discrete forms of the rendering equation

2.1 Standard continuous forms of the rendering equation

We denote with Σ be the collection of surfaces in the scene and with Ω the unit sphere of direction. Let L be the radiance function, E the emitted radiance function (which is non-zero for light sources), and ρ the BRDF modelling the local behaviour of light on the surfaces. Two standard forms of the rendering equations are reported in [33, 9, 30]. The first form is:

$$(2.1)\qquad L(y,\omega) = E(y,\omega) + \int_{\eta\cdot n_y>0} \rho(y,\omega,\eta)L(x(y,\eta,\Sigma),-\eta)(\eta\cdot n_y)d\eta$$

where y and x are points on the surface Σ, ω and η are directions (points on the unit sphere Ω), $x(y,\omega,\Sigma)$ is the first point of Σ hit by a ray from y in direction ω, and n_y is outer unit vector at y normal to Σ. The second form is:

$$(2.2)\qquad L(y,z) = E(y,z) + \int_{x\in\Sigma} \rho(x,y,z)L(x,y)G(x,y)dx$$

where x, y, z are points on Σ, and

$$(2.3)\qquad G(x,y) = \frac{\cos\theta_x \cos\theta_y}{|x-y|^2} V(x,y,\Sigma)$$

where θ_x (resp. θ_y) is the angle formed by the outer normal n_x (resp. n_y) and the line joining x and y, while the function $V(x,y,\Sigma)$ has value 1 if the open segment xy does not intersect Σ, zero otherwise.

[1] Recently I have become aware that many features of the methods and algorithms in [21, 23] have been independently described in [18, 37]. The main difference between [21, 23] and [18, 37] is in the error analysis. The analysis in [21, 23] is mostly concerned with the computation of the form factors and asymptotic bounds are proved for the complexity of computing all form factors in terms of the number of input polygons and an a priori user-controlled threshold to the absolute error on each form factor. In [18, 37] the error analysis correlates the exact solution of the discretized radiosity linear system with the result obtained by an iterative procedure.

2.2 Simulating occlusions by superimposition

In this section we show how to modify the BRDF function describing the local behaviour of light on the surface in order simulate occlusions by superimposition. The main idea is that energy arriving onto a surface from a direction η produces a negative energy flux in direction $-\eta$ so that behind the surface their sum is null. Naturally this annihilation persists when multiple surfaces are considered on the same ray.

From a mathematical point of view this modified BRDF makes radiosity calculations much more similar to BEM for conductors in electrostatics where the superimposition principle is used. We will show that the occlusion effect is achieved on any single line, thus the cancellation effect carries over on integrals of lines.

Using notation of the previous subsection, we start from a standard form of the rendering equation (2.1). Let us call U_+ the halfsphere where $\eta \cdot n_y > 0$ we integrate contribution of incoming light from U_+ since the assumption is that light outside the objects cannot go inside and vice versa, so to describe the balance of light outside the objects, we assume no sources inside them. So ρ is defined and used for $\eta \in U_+$, but also ω is considered and used in U_+ because by default it is assumed that all the light received from outside is re-sent outside. So $\omega \in U_+$. We are thus free to decide the value to associate with directions in the complementary domain U_- and we use this freedom to simulate occlusions by means of a special BRDF defined on U_-.
So we define: for $\omega \in U_-$ and $\eta \in U_+$:

$$\rho(y, \omega, \eta) = -\frac{\delta(\omega - (-\eta))}{|n_y \cdot \eta|}$$

where δ is the Dirac delta function. Thus carrying out easy calculation from (2.1) we obtain $L(y, -\eta) = -L(x, -\eta)$ which shows the negative flux. Note that L is preserved in absolute value through the surface but changes sign.
For $\eta \in U_-$:

$$\rho(y, \omega, \eta) = 0.$$

Intuitively for rays coming from inside an object all contributions cancel off anyway so we do not need to count them.

We have to modify the rendering equation and collect contributions from all the intersection points along a ray, not just the visible one. The new form of the rendering equation is:

$$(2.4)\; L(y, \omega) = E(y, \omega) + \int_{\eta \cdot n_y > 0} \rho'(y, \omega, \eta) \left[\sum_{m=1}^{v} L(x_m(y, \eta), -\eta) \right] |\eta \cdot n_y| d\eta$$

where $x_1 \dots x_v$, is the collection of points of intersection between the ray from y in direction η on the surfaces in Σ. With ρ' we denote the BRDF modified as above.

An alternative way of viewing the superimposition is the following: it corresponds to using formulae of inclusion/exclusion instead of describing explicitly the combinatorial structure of the visible set.

2.3 Derivation of the new discrete form

We start our manipulations from formula (2.4). Let (y, ω) define the oriented line l, and let (y, η) define the oriented line h, then easily $(x, -\eta)$ defines the oriented line $-h$. This notation will be important later to distinguish between the lines and their representation using a coordinate system. We assume now an expansion of the function L into a functional basis. Specifically we consider a covering of the domain $\Sigma \times \Omega$

$$\Sigma \times \Omega = \cup_{i=1}^{n} (\Sigma_i \times \Omega_i)$$

Let $f_{i,j}(\cdot, \cdot)$ be a basis function with compact support in $\Sigma_i \times \Omega_i$ (i.e. it is identically zero in $(\Sigma \times \Omega) \setminus (\Sigma_i \times \Omega_i)$). We express the approximate solution $\tilde{L}$ with $\tilde{L} = \sum_{i,j} a_{i,j} f_{i,j}$ for unknown reals $a_{i,j}$. We now formally multiply equation (2.4) by the weight function $f_{i,j}$ and we compute the weighted residual which is then set to zero. Practically we multiply by $f_{i,j}$ and integrate on the differential surface element around y: dy, for $y \in \Sigma_i$, and by the differential element of the angle $\omega \in \Omega_i$: $d\omega$.

$$\begin{aligned} & f_{i,j}(y, \omega) L(y, \omega) dy d\omega = \\ & f_{i,j}(y, \omega) E(y, \omega) dy d\omega + \\ & \left[\int_{\eta \cdot n(y) \geq 0} \rho'(y, \omega, \eta) [\sum_m L(x_m, -\eta)] |\eta \cdot n(y)| d\eta \right] \\ & f_{i,j}(y, \omega) dy d\omega \end{aligned}$$

next we integrate over the domain $\Sigma \times \Omega$, however due to the choice of support for the functions $f_{i,j}$ this is equivalent to integrating over $\Sigma_i \times \Omega_i$.

$$\begin{aligned} & \int_{y \in \Sigma_i} \int_{\omega \in \Omega_i} f_{i,j}(y, \omega) L(y, \omega) dy d\omega = \\ & \int_{y \in \Sigma_i} \int_{\omega \in \Omega_i} f_{i,j}(y, \omega) E(y, \omega) dy d\omega + \\ & \int_{y \in \Sigma_i} \int_{\omega \in \Omega_i} \int_{\eta \cdot n(y) \geq 0} \\ & \rho'(y, \omega, \eta) [\sum_m L(x_m, -\eta)] |\eta \cdot n(y)| d\eta f_i(y, \omega) dy d\omega \end{aligned}$$

Up to now the derivation has been quite standard. Since $|\eta \cdot n(y)| d\eta dy$ is the differential measure of lines around h, dh, for h meeting Σ_i [27], thus we can rewrite more compactly:

$$\int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} f_{i,j}(y,\omega)L(y,\omega)dyd\omega =$$
$$\int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} f_{i,j}(y,\omega)E(y,\omega)dyd\omega + \tag{2.5}$$
$$\int_{\omega\in\Omega_i}\int_{h\cap\Sigma_i\neq\emptyset} f_{i,j}(y,\omega)\rho'(y,\omega,\eta)[\sum_m L(-h_m)]dhd\omega$$

Where by h_m we denote the pair (x_m,η). Since $L(-h_m)$ does not depend on ω we can exchange the order of integration and take $L(-h_m)$ out of the integration in $d\omega$:

$$\int_{h\cap\Sigma_i\neq\emptyset}[\sum_m L(-h_m)]\left[\int_{\omega\in\Omega_i}\rho'(y,\omega,\eta)f_{i,j}(y,\omega)d\omega\right]dh.$$

We assume that we have chosen the functions ρ' and $f_{i,j}$ in a class for which we are able to integrate analytically in $d\omega$. More precisely we assume the following function to be computable analytically:

$$Q_{i,j}(h) = Q_{i,j}(y,\eta) = \int_{\omega\in\Omega_i} f_{i,j}(y,\omega)\rho'(y,\omega,\eta)d\omega \tag{2.6}$$

Polynomials naturally satisfy this property. From here it is convenient to show two derivations, one specific for a piecewise constant functional basis and one for more general functional basis.

2.4 General functional basis

We assume that for each domain $\Sigma_i \times \Omega_i$ we have either one function f_i or several functions orthonormal functions $f_{i,j}$. In either case we discuss the terms of equation (2.5):

$$\int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} f_{i,j}(y,\omega)L(y,\omega)dyd\omega =$$
$$a_{i,j}\int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} f_{i,j}(y,\omega)^2 dyd\omega$$

For short we denote with $H_{i,j}$ the coefficient of $a_{i,j}$:

$$H_{i,j} = \int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} f_i(y,\omega)^2 dyd\omega,$$

which we suppose to be able to integrate analytically. The emitted energy is weighted by the function $f_{i,j}$:

$$\int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} f_{i,j}(\omega,y)E(y,\omega)dyd\omega = E_{i,j},$$

and we are able to compute $E_{i,j}$ analytically when it is a polynomial. Summarizing:

$$a_{i,j}H_{i,j} = E_{i,j} + \int_{h\cap\Sigma_i\neq\emptyset}[\sum_m L(-h_m)]Q_{i,j}(h)dh$$

Using the expansion of $L(-h)$ we get:

$$a_{i,j}H_{i,j} = E_{i,j} + \int_{h\cap\Sigma_i\neq\emptyset}\left[\sum_{k\neq i,s} a_{k,s}f_{k,s}(-h)\right]Q_{i,j}(h)dh$$

rearranging:

$$a_{i,j}H_{i,j} = E_{i,j} + \sum_{k\neq i,p} a_{k,s}\left[\int_{h\cap\Sigma_i\neq\emptyset} f_{k,s}(-h)Q_{i,j}(h)dh\right]$$

Now we restrict the integration domain to lines for which $f_{k,s}$ is not zero, therefore obtaining the following expression for the coefficient $K_{ij,ks}$ of $a_{k,s}$:

$$K_{ij,ks} = \int_{h\cap\Sigma_i\neq\emptyset} f_{k,s}(-h)Q_{i,j}(h)dh = \int_{h\cap\Sigma_i\neq\emptyset,-h\cap\Sigma_k\neq\emptyset,Dir(-h)\in\Omega_k} f_{k,s}(-h)Q_{i,j}(h)dh \tag{2.7}$$

where $Dir(\cdot)$ denotes the direction of a line. Note that here the function $Q_{i,j}$ carries contribution from the BDRF and from the basis function $f_{i,j}$.

2.5 Piecewise constant basis Let us consider the special case of piecewise constant discretization of L, we drop the subscript j since a unique unit step function is associated with each domain $\Sigma_i \times \Omega_i$. Let us discuss each term of equation (2.5):

$$\int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} L(y,\omega)dyd\omega = a_i|\Sigma_i||\Omega_i|$$

and:

$$\int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} E(y,\omega)dyd\omega = E_i.$$

When the function E is piecewise polynomial we are able to compute analytically the above integral. Summarizing:

$$a_i|\Sigma_i||\Omega_i| = E_i + \int_{h\cap\Sigma_i\neq\emptyset}[\sum_m L(-h_m)]Q_i(h)dh$$

Using the expansion of $L(-h)$ we get:

$$a_i|\Sigma_i||\Omega_i| = E_i + \int_{h\cap\Sigma_i\neq\emptyset} \left[\sum_{k\neq i} a_k f_k(-h)\right] Q_i(h)dh$$

rearranging:

$$a_i|\Sigma_i||\Omega_i| = E_i + \sum_{k\neq i} a_k \left[\int_{h\cap\Sigma_i\neq\emptyset} f_k(-h)Q_i(h)dh\right]$$

Now we restrict the integration domain to lines for which f_k is not zero, obtaining the following expression for the coefficient of a_k:

$$\begin{aligned}&\int_{h\cap\Sigma_i\neq\emptyset} f_k(-h)Q_i(h)dh = \\ (2.8) \qquad &\int_{h\cap\Sigma_i\neq\emptyset,-h\cap\Sigma_k\neq\emptyset,Dir(-h)\in\Omega_k} Q_i(h)dh\end{aligned}$$

where $Dir(\cdot)$ denotes the direction of a line.

In the next section we will show that coefficients (2.8) and (2.7) have a form very similar to expressions for higher order form factors (the corresponding coefficients of radiosity equations) developed in [25]. As a consequence (asymptotically) fast integration algorithms described in [25] can be used for approximating (2.8) and (2.7).

3 Evaluation of the matrix coefficients

In [25] it is shown that the higher order form factor between basis functions $f_{i,j}$ and $f_{k,s}$ supported on triangles T_i and T_k can be represented by the following integrals of lines:

$$(3.9) \qquad C_{ij,ks} = \int_{l,L\cap T_i\neq\emptyset,l\cap T_k\neq\emptyset} f_{ij}(p)f_{ks}(p')dl$$

where $p = l\cap T_1$ and $p' = l\cap T_2$, and dl is the differential measure of lines in 3-space [27]. Next we express a line l as a pair (u,q), where u is the direction of l, and q is the intercept of l with a plane $P(u)$ of normal u containing the origin. In such coordinates $dl = dudq$, therefore:

$$(3.10) \qquad C_{ij,ks} = \int_{u\in\Omega} \left[\int_q f_{ij}(p)f_{ks}(p')dq\right] du.$$

The hypothesis placed on the functions f is that they are polynomial in Cartesian coordinates, since this ensures that fore very fixed direction u the inner integral

$$\int_q f_{ij}(p)f_{ks}(p')dq$$

is computable analytically. Following the same scheme we express $h = (q,u)$ and we obtain the following expression for $K_{ij,ks}$:

$$(3.11)\, K_{ij,ks} = \int_{-u\in\Omega_k} \int_q f_{ks}(-u,q)Q_{ij}(u,q)dqdu$$

To apply the technique in [25] we make the assumption that, for any fixed u, $f_{ks}(-u,q)$ is polynomial in local Cartesian coordinates for q. This assumption is easy to satisfy by using a product polynomial basis. Also $Q_{ij}(u,q)$ needs to satisfy the same condition, which is discussed in the next section.

Finally the domain of the external integration is restricted to direction such that $-u\in\Omega_k$. If we restrict the possible shapes for Ω_k to convex spherical polygons (therefore a shape bounded by arcs of great circles on Ω) we can accommodate the range reduction within the spherical domains considered in [25]. Therefore, using the result in [25] for higher order form factors, it is possible to prove that:

THEOREM 2. *Integrals (2.8) and (2.7) can be computed up to absolute error ϵ in time $O(\log_c^2(1/\epsilon))$ for a constant $c > 1$.*

4 Notes on the conditions imposed on the BDRF

The stronger assumptions used in this derivation seem to be those placed on the BDRF, in particular we assume that the integral in (2.6) can be solved analytically, and that the function $Q_{ij}(q,u)$ is polynomial in the local cartesian coordinates of q for any fixed value of u.

If the BRDF is piecewise polynomial in Cartesian coordinate these conditions are met. However it is possible to enlarge the class of admissible functions. In scenes composed of polygonal surfaces, where each surface patch is of homogeneous material the BRDF is usually independent from the coordinates of points on the patch, thus, if we use a polynomial basis function with separable spatial and angular variables, the second requirement is easily satisfied.

A detailed survey of models for BRDFs used in computer graphics is beyond the scope of the present article and we refer the reader to [33, 9] for sources of bibliographical data. In general, BRDFs can be obtained by interpolation of measured data, scattering simulations, physically-based models (e.g. microfacet theory [40]) or empirical models (e.g. Phong's model [26]). A technique due to Arvo [3] can be used to integrate analytically functions defined on the surface of the unit sphere. This technique is suitable for integrating Phong's BRDF. With somewhat more effort

BRDFs defined in [17] and [20] could be fitted within this technique. More complicated models such as those in [28, 12, 40] probably fail the requirement unless a substantial effort of symbolic integration is used.

5 Conclusions

This note describes a new form of the rendering equation introduced by Kajiya to model the equilibrium of radiant energy exchange under a variety of local shading models including diffuse, glossy and specular reflectance. The new form of the integral equation has the same solution as the original form, however it leads to the simulation of occlusions by superimposition. Thus we avoid approaches to occlusions that are either exact but expensive in terms of computing time or heuristic in nature. Our treatment of occlusions is exact in the sense that it does not introduce any further level of approximation besides those that are standard in many Galerkin discretization method used in applied mathematics.

We address a second problem present in numerical solution of rendering equations, namely the high dimensionality of the integrals involved. Using integral geometric transformations we reduce the standard 6-dimensional integrals to 2-dimensional ones which can be approximated numerically up to high precision using a Gaussian adaptive integration scheme in [25].

The technique presented places mild constraints on the admissible BRDF and on the choice of functional basis. Any polynomial functional basis fits the requirements of the method described in this paper. This freedom of choice can be used to attain further beneficial effects. For example, wavelet basis are good candidates to produce (at least in principle) sparse rendering matrices. The technique is completely general and we believe this technique to be competitive in environments where occlusions are present but do not significantly partition the light exchange among surfaces. Implementation of such a scheme and empirical testing are planned as future work.

References

[1] B. Alpert, G. Beylkin, R. Coifman, and V. Rokhlin. Wavelet-like bases for the fast solution of second-kind integral equations. *SIAM Journal on Scientific Computing*, 14(1):159–184, January 1993.

[2] F. Sillon and G. Drettakis. Feature-based control of visibility error: a multi-resolution clustering algorithm for global illumination. In *Computer Graphics (SIGGRAPH '95 Proceedings)*, pages 145–152, August 1995.

[3] James Arvo. Applications of irradiance tensors to the simulation of non-Lambertian phenomena. In Robert Cook, editor, *SIGGRAPH 95 Conference Proceedings*, Annual Conference Series, pages 335–342. ACM SIGGRAPH, Addison Wesley, August 1995. held in Los Angeles, California, 06-11 August 1995.

[4] James Arvo and David B. Kirk. Particle transport and image synthesis. In Forest Baskett, editor, *Computer Graphics (SIGGRAPH '90 Proceedings)*, volume 24, pages 63–66, August 1990.

[5] James Arvo, Kenneth Torrance, and Brian Smits. A framework for the analysis of error in global illumination algorithms. In Andrew Glassner, editor, *Proceedings of SIGGRAPH '94 (Orlando, Florida, July 24–29, 1994)*, Computer Graphics Proceedings, Annual Conference Series, pages 75–84. ACM SIGGRAPH, ACM Press, July 1994. ISBN 0-89791-667-0.

[6] L. Aupperle and P. Hanrahan. A hierarchical illumination algorithm for surfaces with glossy reflection. In *Computer Graphics Proceedings, Annual Conference Series, 1993*, pages 155–162, 1993.

[7] G. Beylkin, R. Coifman, and V. Rokhlin. Fast wavelet transforms and numerical algorithms I. *Communications on Pure and Applied Mathematics*, XLIV:141–183, 1991. transformation en ondelettes.

[8] Michael F. Cohen and Donald P. Greenberg. The Hemi-Cube: A radiosity solution for complex environments. In B. A. Barsky, editor, *Computer Graphics (SIGGRAPH '85 Proceedings)*, volume 19, pages 31–40, August 1985.

[9] Michael F. Cohen and John R. Wallace. *Radiosity and Realistic Image Synthesis*. Academic Press Professional, San Diego, CA, 1993.

[10] Steven J. Gortler, Peter Schroder, Michael F. Cohen, and Pat Hanrahan. Wavelet radiosity. In *Computer Graphics Proceedings, Annual Conference Series, 1993*, pages 221–230, 1993.

[11] Pat Hanrahan, David Salzman, and Larry Aupperle. A rapid hierarchical radiosity algorithm. In Thomas W. Sederberg, editor, *Computer Graphics (SIGGRAPH '91 Proceedings)*, volume 25, pages 197–206, July 1991.

[12] Xiao D. He, Kenneth E. Torrance, Francois X. Sillion, and Donald P. Greenberg. A comprehensive physical model for light reflection. In Thomas W. Sederberg, editor, *Computer Graphics (SIGGRAPH '91 Proceedings)*, volume 25, pages 175–186, July 1991.

[13] P.S. Heckbert. Finite element methods for radiosity. Global illumination course notes Siggraph 93. Available at www.cs.cmu.edu/afs/cs/user/ph/www/heckbert.html.

[14] P.S. Heckbert. Introduction to finite element methods. Global illumination course notes Siggraph 93. Available at www.cs.cmu.edu/afs/cs/user/ph/www/heckbert.html.

[15] David S. Immel, Michael F. Cohen, and Donald P. Greenberg. A radiosity method for non-diffuse environments. In David C. Evans and Russell J. Athay, editors, *Computer Graphics (SIGGRAPH '86 Proceedings)*, volume 20, pages 133–142, August 1986.

[16] James T. Kajiya. The rendering equation. In David C. Evans and Russell J. Athay, editors, *Computer Graphics (SIGGRAPH '86 Proceedings)*, volume 20, pages

143–150, August 1986.

[17] Eric P. F. Lafortune, Sing-Choong Foo, Kenneth E. Torrance, and Donald P. Greenberg. Non-linear approximation of reflectance functions. In Turner Whitted, editor, *SIGGRAPH 97 Conference Proceedings*, Annual Conference Series, pages 117–126. ACM SIGGRAPH, Addison Wesley, August 1997. ISBN 0-89791-896-7.

[18] L. Neumann. Monte Carlo radiosity. *Computing*, 55:23–42, 1995.

[19] Laszlo Neumann and Attila Neumann. Radiosity and hybrid methods. *ACM Transactions on Graphics*, 14(3):233–265, July 1995.

[20] Michael Oren and Shree K. Nayar. Generalization of Lambert's reflectance model. In Andrew Glassner, editor, *Proceedings of SIGGRAPH '94 (Orlando, Florida, July 24–29, 1994)*, Computer Graphics Proceedings, Annual Conference Series, pages 239–246. ACM SIGGRAPH, ACM Press, July 1994. ISBN 0-89791-667-0.

[21] M. Pellegrini. Monte Carlo approximation of form factors with error bounded a priori. In *11th ACM Symposium on Computational Geometry*, June 1995.

[22] M. Pellegrini. Electrostatic fields without singularities: Theory and algorithms. In *Proceedings of the 7th ACM-SIAM Symposium on Discrete Algorithms*, pages 184–191, 1996. Full version to appear in *Journal of the ACM*.

[23] M. Pellegrini. Monte Carlo approximation of form factors with error bounded a priori. *Discrete & Computational Geometry*, 17:319–337, 1997.

[24] M. Pellegrini. Ray shooting and lines in space. In Jacob E. Goodman and Joseph O'Rourke, editors, *CRC Handbook of Discrete and Computational Geometry*, pages 599–614. CRC Press, 1997.

[25] M. Pellegrini. On numerical approximation of higher order form factors. Technical Report TR IMC B4-98-01, Istituto di Matematica Computazionale del CNR, Via Santa Maria 46, Pisa, Italy, January 1998. http://www.imc.pi.cnr.it/ marcop/papiri/hoff-paper.ps.

[26] Bui-T. Phong. Illumination for computer generated pictures. *Communications of the ACM*, 18(6):311–317, June 1975.

[27] L. A. Santaló. *Integral Geometry and Geometric Probability*. Addison-Wesley, Reading, MA, 1976.

[28] Christophe Schlick. A customizable reflectance model for everyday rendering. In Michael F. Cohen, Claude Puech, and Francois Sillion, editors, *Fourth Eurographics Workshop on Rendering*, pages 73–84. Eurographics, June 1993. held in Paris, France, 14–16 June 1993.

[29] Peter Schroder, Steven J. Gortler, Michael F. Cohen, and Pat Hanrahan. Wavelet projections for radiosity. *Computer Graphics Forum*, 13(2):141–151, June 1994.

[30] Peter Schroder and Pat Hanrahan. Wavelet methods for radiance computations. In *Fifth Eurographics Workshop on Rendering*, pages 303–311, Darmstadt, Germany, June 1994.

[31] Min-Zhi Shao, Qun-Sheng Peng, and You-Dong Liang. A new radiosity approach by procedural refinements for realistic image synthesis. In John Dill, editor, *Computer Graphics (SIGGRAPH '88 Proceedings)*, volume 22, pages 93–101, August 1988.

[32] Peter Shirley. A ray tracing method for illumination calculation in diffuse-specular scenes. In *Proceedings of Graphics Interface '90*, pages 205–212, May 1990.

[33] François Sillion and Claude Puech. *Radiosity and Global Illumination*. Morgan Kaufmann, San Francisco, 1994. excellent coverage of radiosity and global illumination algorithms.

[34] Francois X. Sillion, James R. Arvo, Stephen H. Westin, and Donald P. Greenberg. A global illumination solution for general reflectance distributions. In Thomas W. Sederberg, editor, *Computer Graphics (SIGGRAPH '91 Proceedings)*, volume 25, pages 187–196, July 1991.

[35] B. Smits, J. Arvo, and D. Greenberg. A clustering algorithm for radiosity in complex environments. In *Proceedings of SIGGRAPH '94*, pages 435–442, 1994. In Computer Graphics.

[36] C. Soler and F. Sillion. Fast calculation of soft shadow textures using convolution. To appear in Siggraph 98 Conference Proceedings, 1998.

[37] L. Szirmay-Kalos, T. Foris, L. Neumann, and B. Csebfalvi. An analysis of Quasi-Monte Carlo integration applied to the transillumination radiosity method. *Computer Graphics Forum*, 16(3):271–281, 1997.

[38] Seth Teller and Pat Hanrahan. Global visibility algorithms for illumination computations. In *Computer Graphics Proceedings, Annual Conference Series, 1993*, pages 239–246, 1993.

[39] Seth J. Teller and Carlo H. Séquin. Visibility preprocessing for interactive walkthroughs. In Thomas W. Sederberg, editor, *Computer Graphics (SIGGRAPH '91 Proceedings)*, volume 25, pages 61–69, July 1991.

[40] K. E. Torrance and E. M. Sparrow. Theory for off-specular reflection from roughened surfaces. *Journal of Optical Society of America*, 57(9), 1967.

[41] Roy Troutman and Nelson L. Max. Radiosity algorithms using higher order finite element methods. In *Computer Graphics Proceedings, Annual Conference Series, 1993*, pages 209–212, 1993.

[42] Eric Veach and Leonidas J. Guibas. Metropolis light transport. In Turner Whitted, editor, *SIGGRAPH 97 Conference Proceedings*, Annual Conference Series, pages 65–76. ACM SIGGRAPH, Addison Wesley, August 1997. ISBN 0-89791-896-7.

[43] Andrew J. Willmott and Paul S. Heckbert. An empirical comparison of progressive and wavelet radiosity. In Julie Dorsey and Philipp Slusallek, editors, *Eurographics Rendering Workshop 1997*, pages 175–186, New York City, NY, June 1997. Eurographics, Springer Wein. ISBN 3-211-83001-4.

[44] Harold R. Zatz. Galerkin radiosity: A higher order solution method for global illumination. In *Computer Graphics Proceedings, Annual Conference Series, 1993*, pages 213–220, 1993.